Thompson's group T, groups acting on the circle and Poisson boundaries

based on joint work with **Martín Gilabert Vio** and **Cosmas Kravaris**

Eduardo Silva (eduardo.silva@uni-muenster.de) YGGT XIII – University of Copenhagen, 2025

University of Münster, Germany

Harmonic functions on groups

Let G be a countable group and let μ be a probability measure on G. A function $f : G \to \mathbb{R}$ is called **μ-harmonic** if $f(g) = \sum_{h \in G} f(gh)\mu(h)$ for all $g \in G$. The **Poisson boundary** (B, ν) of (G, μ) is a probability G-space such that ν **is μ-stationary** (i.e. $\nu = \mu * \nu$), and provides an isomorphism of Banach algebras

$$L^\infty(B,\nu) \to \{f : G \to \mathbb{R} \mid f \text{ bounded and } \mu\text{-harmonic}\}$$
$$F \mapsto \left(f(g) = \int_B F(gx)d\nu(x), \text{ for } g \in G\right)$$

Problem: Describe the Poisson boundary in terms of the geometric properties of G.

One can often identify **G-equivariant quotients of** (B, ν), called **μ-boundaries**.

Knowing a μ-boundary corresponds to finding a subspace of bounded μ-harmonic functions. Saying that it is the Poisson boundary means that there are none of them missing.

Examples of Poisson boundaries

Gromov-hyperbolic groups. Let G be a non-elementary Gromov hyperbolic group, and denote by ∂G its Gromov boundary. Then for any non-elementary $\mu \in \mathrm{Prob}(G)$ there is a unique μ-stationary probability measure ν on ∂G. If $H(\mu) < \infty$, then $(\partial G, \nu)$ is the Poisson boundary of (G, μ) (Kaimanovich '94, Chawla-Forghani-Frisch-Tiozzo '22, and many more).

Wreath products. Consider the lamplighter groups $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d := \left(\bigoplus_{\mathbb{Z}^d} \mathbb{Z}/2\mathbb{Z}\right) \rtimes \mathbb{Z}^d$, $d \geq 3$. Let $\mu \in \mathrm{Prob}(\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d)$ be a non-degenerate finitely supported probability measure. Then there is a μ-stationary probability measure ν on $\prod_{\mathbb{Z}^d} \mathbb{Z}/2\mathbb{Z}$ such that $(\prod_{\mathbb{Z}^d} \mathbb{Z}/2\mathbb{Z}, \nu)$ is the Poisson boundary of $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$ (Erschler '11, Lyons-Peres '21).

Groups of homeomorphisms of the circle

The action $G \curvearrowright S^1$ is called **proximal** if for every proper interval $I \subset S^1$ and every $\epsilon > 0$ there is $g \in G$ with $\mathrm{diam}(g(I)) < \epsilon$.

Theorem [Deroin-Kleptsyn-Navas '07]

Let $G \curvearrowright S^1$ by orientation-preserving homeomorphisms with no invariant probability measure on S^1, and let $\mu \in \mathrm{Prob}(G)$ be non-degenerate. Suppose that $G \curvearrowright S^1$ is proximal. Then there is a **unique μ-stationary probability measure** ν on S^1, and (S^1, ν) is a μ-boundary of G.

The proof goes as follows: one shows that for almost every sample path $\mathbf{w} = (w_n)_{n \geq 0} \in G^{\mathbb{N}}$ of the μ-random walk on G there exists a point $\xi(\mathbf{w}) \in S^1$ such that $\lim_{n\to\infty}(w_n)_*\nu = \delta_{\xi(\mathbf{w})}$ in the weak-$*$ topology. The measure ν is the distribution of $\xi(\mathbf{w})$ on S^1.

Theorem [Deroin '13]

Suppose furthermore that the action $G \curvearrowright S^1$ is **strongly discrete** and sufficiently regular, and that μ is finitely supported. Then (S^1, ν) is the Poisson boundary of (G, μ).

This is satisfied in particular by cocompact lattice in $\mathrm{PSL}_2(\mathbb{R})$. The groups covered by the above result fall within a family that is conjectured to be composed only of Gromov-hyperbolic groups, and hence their Poisson boundaries could alternatively be described using their Gromov boundaries.

Question [Deroin '13, Navas '17]: Is (S^1, ν) always the Poisson boundary of (G, μ)?

Main Theorem [Gilabert - Kravaris - S. '25]

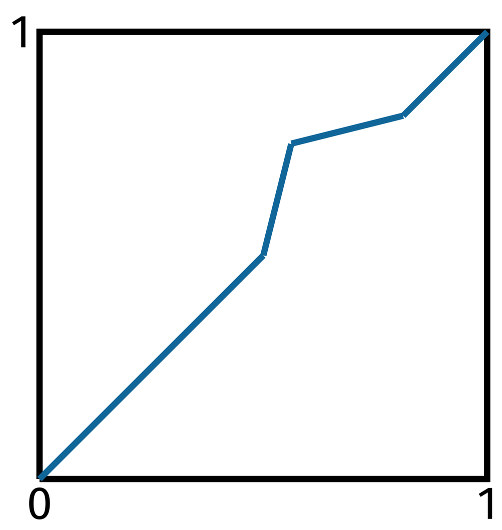
Let $G \leq \mathrm{Homeo}_+(S^1)$ be a countable group acting proximally, minimally and topologically non-freely on S^1. Let μ be a non-degenerate probability measure on G with $-\sum_{g\in G}\mu(g)\log(\mu(g)) < \infty$. Then (S^1, ν) is not the Poisson boundary of (G, μ).

This applies in particular to Thompson's group T, the group of dyadic piecewise affine homeomorphisms of the circle.

Our main theorem is related to the well-known open problem on whether Thompson's group F, the group of dyadic piecewise affine homeomorphisms of the interval $[0, 1]$, is amenable. Indeed, the action of a countable group G on its Poisson boundary $(\partial_\mu G, \nu)$ is amenable, and hence for ν-almost every $x \in \partial_\mu G$ the stabilizer subgroup $G_x \leq G$ is amenable. If the circle were the Poisson boundary of T then we would conclude that F is amenable, since for each $x \in S^1$ the stabilizer $T_x \leq T$ contains a copy of F. Our theorem implies that this strategy does not work for μ with finite entropy.

The proof for Thompson's group T and finitely supported μ

- Thompson's group T is the group of **dyadic piecewise affine homeomorphisms of the circle**: that is, T is the group of orientation-preserving homeomorphisms $g : S^1 \to S^1$ such that the derivative of g is defined outside a finite subset of the dyadic rationals $\mathbb{Z}[1/2]/\mathbb{Z}$ and takes values in $\{2^k\}_{k\in\mathbb{Z}}$.
- For each $g \in T$, define a finitely supported function $C_g \colon \mathbb{Z}[1/2]/\mathbb{Z} \to \mathbb{R}$ by setting
$$C_g(x) = \log_2\left((g^{-1})'(x^+)\right) - \log_2\left((g^{-1})'(x^-)\right), \text{ for } x \in \mathbb{Z}[1/2]/\mathbb{Z},$$
where $(g^{-1})'(x^+)$ (resp. $(g^{-1})'(x^-)$) is the left (resp. right) derivative of g^{-1} at x. That is, **$C_g(x)$ is the derivative jump of g^{-1} at x.**
- Denote the set of all (not necessarily finitely supported) functions $\mathbb{Z}[1/2]/\mathbb{Z} \to \mathbb{R}$ by $\mathbb{R}^{\mathbb{Z}[1/2]/\mathbb{Z}}$.
- For almost every trajectory $\mathbf{w} = (w_n)_n \in T^{\mathbb{N}}$ of the μ-random walk, the configurations $(C_{w_n})_{n\geq 0}$ converge pointwise to a map $C_\infty(\mathbf{w}) \in \mathbb{R}^{\mathbb{Z}[1/2]/\mathbb{Z}}$. The hitting measure λ is a μ-stationary prob. measure on $\mathbb{Z}^{\mathbb{Z}[1/2]/\mathbb{Z}}$ such that **the space $(\mathbb{Z}^{\mathbb{Z}[1/2]/\mathbb{Z}}, \lambda)$ is a μ-boundary of T.**
- Since the measure λ is nontrivial, there exists $y \in \mathbb{Z}[1/2]/\mathbb{Z}$ and $k \in \mathbb{Z}$ such that $f \colon T \to [0, 1]$ defined by
$$f(g) = \mathbb{P}_g\left[\mathbf{w} \in T^{\mathbb{N}} \mid C_\infty(\mathbf{w})(y) = k\right], \text{ for } g \in T$$
satisfies $f(e_T) > 0$. The function f is bounded and μ-harmonic.
- There exists a sequence $\{g_n\}_{n\geq 0} \subseteq T$ such that $\mathrm{supp}(g_n)$ are closed intervals containing y and such that $\mathrm{diam}(\mathrm{supp}(g_n)) \xrightarrow[n\to\infty]{} 0$ and $f(g_n) \xrightarrow[n\to\infty]{} 0$.
Indeed, one constructs a sequence such that:
  - $g_n(y) = y$,
  - $\mathrm{supp}(g_n)$ is a dyadic interval containing y and of length $2^{-n} + 2^{-2n}$, and
  - the derivative jump of g_n at y is equal to 2^n.



Clearly $\mathrm{diam}(\mathrm{supp}(g_n)) \xrightarrow[n\to\infty]{} 0$. Moreover, if $g \in T$ fixes y we have
$$f(g) = \mathbb{P}_g\left[C_\infty(\mathbf{w})(y) = k\right] = \mathbb{P}\left[C_\infty(\mathbf{w})(y) = k - \log_2(g')^+(y) + \log_2(g')^-(y)\right]$$
so that in particular $f(g_n) = \mathbb{P}\left[C_\infty(\mathbf{w})(y) = k - n\right] \xrightarrow[n\to\infty]{} 0$.
- If (S^1, ν) were the Poisson boundary of (T, μ), then there would exist $h \in L^\infty(S^1, \nu)$ such that
$$f(g) = \int_{S^1} h(gx)d\nu(x), \text{ for all } g \in T.$$
Set $I_n = \mathrm{supp}(g_n)$ for each $n \geq 1$. The equality
$$f(g_n) = \int_{S^1} h(g_n x)d\nu(x) = \int_{S^1\setminus I_n} h(g_n x)d\nu(x) + \int_{I_n} h(g_n x)d\nu(x)$$
and the fact that ν is non-atomic imply that $\int_{S^1\setminus I_n} h(x)d\nu(x) = \int_{S_1\setminus I_n} h(g_n x)d\nu(x) \xrightarrow[n\to\infty]{} 0$. This would imply that $f(e_T) = \int_{S^1} h(x)d\nu(x) = 0$, which is a contradiction.

⚠ **This approach only works for groups of piecewise affine transformations of S^1. The general proof is based on conditional entropy techniques (cf. Kaimanovich, Erschler).**