CONTINUITY OF ASYMPTOTIC ENTROPY OF RANDOM WALKS ON GROUPS

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Abstract. These are notes for my talk at the workshop Groups and Dynamics: Topology, Measure, and Borel Structure in Oberwolfach, 13-17 April 2026. The main objective is to present two continuity results that I proved for the asymptotic entropy, seen as a function on the space of finite entropy probability measures on a countable group.

1. Introduction and motivation

Let μ be a probability measure on a countable group G. A function $f : G \to \mathbb{R}$ is called μ-harmonic if $f(g) = \sum_{h\in G} f(gh)\mu(h)$ for all $g \in G$. The space $H^\infty(G,\mu)$ of bounded μ-harmonic functions is a nice space of functions on which the group acts and which we can use to study the geometry of G. We care about the following vague question:

Question 1.1 (vague)**.** *Let G be a countable group. How "sensitive" is the space $H^\infty(G,\mu)$ of bounded μ-harmonic functions to variations in μ?*

Remark 1.2. In the continuous setting, one has the following result due to Furstenberg '60s: take a connected semisimple Lie group G with finite center and no compact factor. Let K be a maximal compact subgroup and P a minimal parabolic subgroup of G. Think $G = \mathrm{SL}_n(\mathbb{R})$, take $K = SO_n(\mathbb{R})$ and $P \leqslant G$ the subgroup of upper triangular matrices. Let μ be a probability measure that is spherical: absolutely continuous with respect to the Haar measure, and bi-K-invariant. Then

$$H^\infty(G,\mu) \cong L^\infty(G/P, m),$$

where m is the unique K-invariant measure on the G/P. Note that the right-hand side does not depend on μ!

In the discrete setting of course we cannot expect to have *exactly* the same space of bounded harmonic functions for two different measures. The way that we will quantify the sensitivity of $H^\infty(G,\mu)$ is through asymptotic entropy.

Definition 1.3. Let μ a non-degenerate probability measure on a countable group G, and such that $H(\mu) \coloneqq -\sum_{g\in G} \mu(g)\log\mu(g) < \infty$, so that the *asymptotic entropy*

$$h(\mu) \coloneqq \lim_{n\to\infty} \frac{H(\mu^{*n})}{n}$$

is well-defined and finite. This quantity was first introduced by Avez in the early 70's.

Theorem 1.4 (Derriennic '80, Kaimanovich-Vershik '83)**.**

$$H^\infty(G,\mu) \text{ contains non-constant functions} \iff h(\mu) > 0.$$

From now on in the talk I will concentrate in the next question.

Question 1.5 (concrete)**.** *Let G be a countable group, $(\mu_i)_i, \mu$ probability measures on G with finite entropy. Suppose that*

- $\lim_{i\to\infty} \mu_i(g) = \mu(g)$ *for all $g \in G$, and*
- $\lim_{i\to\infty} H(\mu_i) = H(\mu)$.

Is it true that $\lim_{i\to\infty} h(\mu_i) = h(\mu)$*?*

I will denote the two convergences above as $\mu_i \xrightarrow[i\to\infty]{} \mu$ to save space.

Some history and motivation:

- Historically, the first example of discontinuity is due to Kaimanovich-Vershik '83. They exhibited measures with finite entropy and non-trivial Poisson boundary on the group $\mathrm{FSym}(\mathbb{Z})$ of finitary permutations of $\mathbb{Z}$. Now of course we know that one can always do this on a non-trivial group with an ICC quotient, thanks to Frisch-Hartman-Tamuz-Vahidi Ferdowsi '19.
- For random walks with a finite first moment on a finitely generated group, another related quantity is the asymptotic drift $\ell(\mu) := \lim_{n\to\infty} E[|w_n|]/n$ of the random walk $(w_n)_n$ with respect to a word metric. In many classes of groups, the ratio $h(\mu)/\ell(\mu)$ coincides with the Hausdorff dimension of the hitting measure of the μ-random walk on a geometric boundary of the group G. This is the case for free groups (Kaimanovich '98) and hyperbolic groups (Le Prince '08).

2. The results

Theorem 2.1 (S. '25). *Let G, $(\mu_i)_i, \mu$ non-degenerate as above with $\mu_i \to \mu$. Suppose there is a Polish G-space X and $(\nu_i)_i, \nu \in \mathrm{Prob}(X)$ such that (X, ν_i) is the Poisson boundary of (G, μ_i) for each i, and such that (X, ν) is the Poisson boundary of (G, μ).*

- *If ν_i converges weakly to ν, then $\lim_{i\to\infty} h(\mu_i) = h(\mu)$.*
- *Suppose that X is further supposed to be compact and uniquely μ-stationary. Then it always holds that ν_i converges weakly to ν.*

Hence, the difficulties associated with showing the continuity of entropy are replaced by the difficulties of identifying the Poisson boundary of the random walks on the group. The above result proves continuity in the next contexts:

- Acylindrically hyperbolic groups. In this case, the continuity of asymptotic entropy had been established previously, with different sets of hypotheses on the measures, by Erschler-Kaimanovich, Gouëzel-Matheus-Macourant, and Choi. All of their proofs are rather technical and rely on the moment hypotheses that they consider (except Choi's proof, which has no moment hypotheses).
- $\mathrm{SL}_d(\mathbb{Z})$ with $d \geq 3$. In this case, the continuity of asymptotic entropy is completely new, even if you only consider probability measures supported on a fix generating set of the group.

I refer to the citations on the paper for the references of identifications of Poisson boundaries that are used for the above applications, in addition to the recent preprint of Chawla-Frisch-Forghani-Tiozzo '25 for lattices in semisimple Lie groups.

The second theorem I want to speak about is the following.

Theorem 2.2 (S. '25). *Let A be a countable group and consider the wreath product $G = A \wr \mathbb{Z}^d := \bigoplus_{\mathbb{Z}^d} A \rtimes \mathbb{Z}^d$ with $d \geq 3$. Then asymptotic entropy is continuous among all non-deg. probability measures with finite entropy. More generally, instead of non-deg., you can just ask that all probability measures have a projection to $\mathbb{Z}^d$ with support of at least cubic growth.*

This theorem is particularly interesting since at the generality of finite entropy measures, there is no identification of the Poisson boundary for $A \wr \mathbb{Z}^d$. Further, work in progress of Josh Frisch and I shows that there cannot exist a "universal realization". We can prove that there does not exist a Polish space X on which the Poisson boundary of all finite entropy non-deg. random walks on $A \wr \mathbb{Z}^d$ is modeled.

One should also remark that the theorem is false is $d = 1, 2$, and this goes back to examples of discontinuity of asymptotic entropy on wreath products due to Erschler.

You can think of entropy on a wreath product as bits of information being encoded by the lamps visited by the walker. Hence, the entropy is directly related to the asymptotic amount of distinct positions visited by the random walk on the base group. A key step in the proof of the above theorem is showing that in general, on groups with at least cubic growth, this quantity is continuous.

3. Some discussion and next questions

I want to spend the last minutes of the talk motivating a situation in which none of the theorems apply.

Let $G \leqslant \mathrm{Homeo}_+(S^1)$ so that $G \curvearrowright S^1$ is minimal and proximal. Deroin, Kleptsyn and Navas '07 showed that for any $\mu \in \mathrm{Prob}(G)$ there is a unique μ-stationary probability measure ν on S^1, and (S^1, ν) is a μ-boundary, i.e., a G-equivariant quotient of the Poisson boundary of (G, μ). If (S^1, ν) were to be the Poisson boundary, then one could apply our first theorem to conclude continuity of entropy. This is the case, for example, for cocompact lattices in $\mathrm{PSL}_2(\mathbb{R})$. However, there is a large class of groups for which this approach will fail.

Theorem 3.1 (Gilabert Vio, Kravaris, S. '25)**.** *In the above context, suppose further that $G \curvearrowright S^1$ is topologically non-free, i.e., there is a non-trivial element that acts trivially on an open subset of S^1. Then for any $\mu \in \mathrm{Prob}(G)$ with $H(\mu) < \infty$, the space (S^1, ν) is **not** the Poisson boundary of (G, μ).*

This result covers Thompson's group T. This motivates the following question.

Question 3.2. *Is asymptotic entropy continuous in Thompson's group T?*

This would be interesting even when restricting asymptotic entropy to the space of probability measures with support exactly equal to a given fixed finite symmetric generating set of T.

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