Stationary boundaries on the space of amenable subgroups

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C^*-simplicity and the amenable radical

$$C_r^*(G) = \overline{\text{span}\{\lambda_g : g \in G\}}^{\|\cdot\|} \subseteq \mathcal{B}(\ell^2(G)), \quad \lambda = \text{left reg. rep.}$$

- G is C^*-simple if $C_r^*(G)$ has no non-zero proper closed two-sided ideals.
- G has the unique trace property if $\tau(a) = \langle a\delta_e, \delta_e \rangle$ is the only tracial state on $C_r^*(G)$.
- The amenable radical $R_{\mathrm{am}}(G)$ of G is its largest amenable normal subgroup.

Facts:

$$\text{C*-simple} \Longrightarrow R_{\mathrm{am}}(G) = \{e_G\}.$$

$$\text{UTP} \Longrightarrow R_{\mathrm{am}}(G) = \{e_G\}.$$

Le Boudec '16: Non-C*-simple groups with $R_{\mathrm{am}}(G) = \{e_G\}$.

Characterization through Furstenberg boundary

The **Furstenberg boundary** $\partial_F G$ is the universal G-boundary: every compact, minimal, and strongly proximal G-space is a G-equivariant quotient of $\partial_F G$.

$$\underline{\text{Furman '03}} : \ker\big(G \curvearrowright \partial_F G\big) = \mathrm{Rad}_{\mathrm{am}}(G).$$

[Kalantar–Kennedy '17, Breuillard–Kalantar–Kennedy–Ozawa '17]

G is C^*-simple $\iff$ $G \curvearrowright \partial_F G$ is free,

G has the UTP $\iff$ $\mathrm{Rad}_{\mathrm{am}}(G) = \{e_G\} \iff G \curvearrowright \partial_F G$ is faithful.

C*-simplicity and random subgroups

- Chabauty space: $G \curvearrowright \mathrm{Sub}(G)$ by conjugation.
- $\mathrm{Sub}_{\mathrm{am}}(G) = \{H \leqslant G : H \text{ is amenable}\}$ is closed and G-invariant.

Bader–Duchesne–Lécureux '16 + BKKO '17: G has a unique trace if and only if $\delta_{\{e_G\}}$ is the unique invariant measure on $\mathrm{Sub}_{\mathrm{am}}(G)$.

Hartman–Kalantar '23

G is C*-simple if and only if there **exists** a non-degenerate $\mu \in \mathrm{Prob}(G)$ for which $\delta_{\{e_G\}}$ is the unique μ-stationary measure on $\mathrm{Sub}_{\mathrm{am}}(G)$.

Many classes satisfy "for **every** μ", e.g. , free groups, linear groups, etc...

Question

Which (C*-simple) groups admit a non-trivial amenable SRS?

Hydration break

Boundary SRSs

Fix $\mu \in \mathrm{Prob}(G)$.

- A measure $\eta \in \mathrm{Prob}\left(\mathrm{Sub}(G)\right)$ is μ-stationary if
$$\eta = \sum_{g \in G} \mu(g)\, g_* \eta.$$
The measure η is called a μ-**stationary random subgroup** (μ-SRS). Stationary measures always exist due to compactness (unlike invariant measures).
- Denote by $w_n = g_1 \cdots g_n$, $n \geq 1$, the right μ-random walk on G. Then the weak*-limit $\lim_{n\to\infty}(w_n)_*\eta$ exists a.s.

Definition: a μ-SRS η is a boundary μ-SRS if $\lim_{n\to\infty}(w_n)_*\eta$ is a.s. a Dirac mass.

Equivalently, if $(\mathrm{Sub}(G), \eta)$ is a G-equivariant quotient of the Poisson boundary of (G, μ).

The existence theorem

For finite $Q \subseteq G$ and $H \leqslant G$ define

$$\mathcal{U}_Q(H) = \{L \in \mathrm{Sub}(G) : L \cap Q = H \cap Q\}.$$

Recurrent orbits property

Say that $H \neq \delta_{\{e_G\}}$ has *recurrent orbits* if for every finite $Q, Z \subseteq G$ there is $b \in G$ such that

$$(bz) \cdot H,\ (b^{-1}z) \cdot H \in \mathcal{U}_Q(H) \ \text{ for every } z \in Z.$$

Theorem [Cascioli – Gilabert Vio – S. '26]

Suppose $H \leqslant G$ is non-trivial and has recurrent orbits. Then there exists a symmetric $\mu \in \mathrm{Prob}(G)$ with $\mathrm{supp}(\mu) = G$ and finite entropy such that

$$w_n H w_n^{-1} \xrightarrow[n\to\infty]{\text{a.s.}} K \neq \delta_{\{e_G\}}.$$

If H is amenable, then $\eta = \mathrm{Law}(K)$ is an amenable boundary μ-SRS.

Frisch-Vigdorovich '26+: This condition is also **necessary**.

Wreath products

Let A, B be countable groups. Consider their wreath product

$$A \wr B := \bigoplus_B A \rtimes B.$$

Fact: If A is C*-simple then so is $A \wr B$.

Corollary

Consider the group

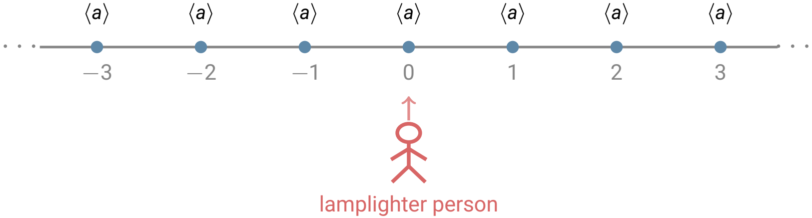
$$G = F_2 \wr \mathbb{Z}^3.$$

Then G is C*-simple and yet **any** $\mu \in \mathrm{Prob}(G)$ **with finite 1st moment (in particular, finite support) admits a non-trivial amenable boundary** μ**-SRS.**

I.e., the Hartman-Kalantar measures **must** have infinite first moment!

Start with $\langle a \rangle$ at every lamp

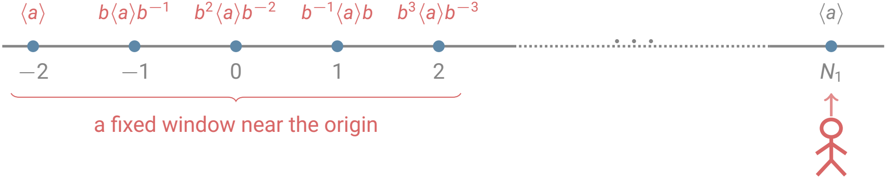
$$G = F_2 \wr \mathbb{Z}, \qquad F_2 = \langle a, b \rangle, \qquad H = \bigoplus_{k \in \mathbb{Z}} \langle a \rangle_k .$$



Every site carries the same cyclic subgroup, and the lamplighter starts at the origin.

Later: the lamplighter modifies lamps and moves

$$w_{n_1} = (\varphi_{n_1}, N_1), \qquad N_1 \gg 1.$$

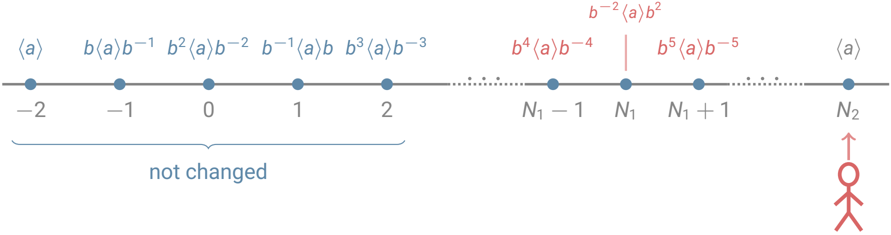


At the site k, the lamp value replaces $\langle a \rangle$ by

$$\varphi_{n_1}(k)\langle a \rangle \varphi_{n_1}(k)^{-1}.$$

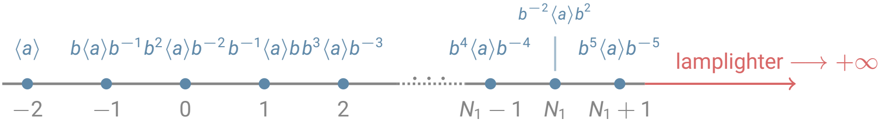
Much later: lamps near the origin stabilize

$$w_{n_2} = (\varphi_{n_2}, N_2), \qquad N_2 \gg N_1 \gg 1.$$



already stabilized **possibly changed farther along the walk**

To infinity

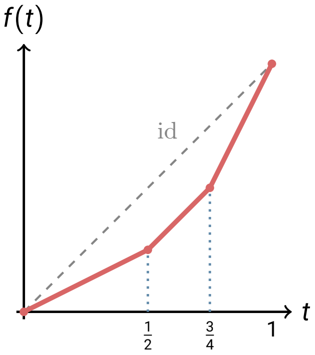


For every fixed $k \in \mathbb{Z}$, the value $\varphi_n(k)$ is eventually constant. Therefore

$$w_n H w_n^{-1} \longrightarrow H_\infty := \bigoplus_{k\in\mathbb{Z}} \varphi_\infty(k)\langle a\rangle\varphi_\infty(k)^{-1}.$$

Thompson's group F consists of the orientation-preserving, piecewise-linear homeomorphisms $f\colon [0,1] \longrightarrow [0,1]$ with finitely many breakpoints, such that

$$\mathrm{Break}(f) \subseteq \mathbb{Z}[1/2], \qquad f'(t) \in 2^{\mathbb{Z}}.$$



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Corollary

There is a symmetric, non-degenerate, finite-entropy μ on Thompson's group F such that $\mathrm{Sub}_{\mathrm{ab}}(F)$ supports a non-atomic μ-boundary.

Matte Bon - Le Boudec '18: F is non-amenable if and only if it is C*-simple.

Boundary SRSs and normalish subgroups

Theorem [Cascioli – Gilabert Vio – S. '26]

Let η be a boundary SRS on $\mathrm{Sub}(G)$ that is not a Dirac mass on a finite normal subgroup. Then η-almost every $H \leqslant G$ is **normalish**:

for every finite subset $Z \subseteq G$ the intersection $\bigcap_{z \in Z} zHz^{-1}$ is infinite.

Corollary: all Poisson boundaries are essentially free

If G has no non-trivial finite normal subgroup and no amenable normalish subgroups, then for **every** non-degenerate μ the action $G \curvearrowright \mathrm{PB}(G, \mu)$ is essentially free.

Applies to: groups with positive first ℓ^2-Betti number, acylindrically hyperbolic groups, linear groups, etc... with trivial amenable radical.