Dead ends on wreath products and lamplighter groups

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Fix a group G and a finite generating set S. Denote by $|\cdot|_S$ the associated word length.

Definition

An element $g \in G$ is called a dead end of depth $\geq M$ *if for every $s_1, \ldots, s_M \in S \cup S^{-1} \cup \{e_G\}$,*

$$|g \cdot s_1 \cdots s_M|_S \leq |g|_S.$$

Example

- $(\mathbb{Z}, \{1\})$ *does not have dead ends.*
- $(\mathbb{Z}, \{2, 3\})$ *has* 1 *and* -1 *as dead ends (of depth* 1*).*

Definition

We say that (G, S) *has unbounded depth if for any* $n \in \mathbb{N}$*, there exists a dead end of depth* $\geq n$*. Otherwise, we say that* (G, S) *has uniformly bounded depth.*

Groups with unbounded depth

Unif. bounded depth for all gensets

Hyperbolic groups [Bogopolski '97]

Abelian groups [Šunić '08, Lehnert '09]

Groups with ≥ 2 ends [Lehnert '09]

Any (G,S) with a regular language of geodesics [Warshall '10]

Virtually abelian groups [Warshall '10]

Unbounded depth for some gensets

The lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ [Cleary & Taback '05]

Many groups of the form $K \rtimes \mathbb{Z}$ with K abelian [Warshall '08]
(in particular Baumslag-Solitar groups $\mathrm{BS}(1, n)$, $n \geq 2$.)

Houghton's group $H_2 = \mathrm{Sym}(\mathbb{Z}) \rtimes \mathbb{Z}$ [Lehnert '09]

Unbounded depth for all gensets

The discrete Heisenberg group [Warshall '11]

Wreath products

The wreath product of A and B is

$$A \wr B := \bigoplus_B A \rtimes B,$$

where $\bigoplus_B A = \{f : B \to A \mid f \text{ of finite support}\}$ and B acts by translations on $f \in \bigoplus_B A$:

$$(b \cdot f)(x) = f(b^{-1}x), \; x, b \in B.$$

Lamplighter interpretation

Multiplying $(f, x) \in A \wr B$ on the right by elements of A changes the lamp configuration f at the current position x, while multiplying by elements of b changes said current position.

Word length on wreath products

Given finite gensets S_A, S_B of A, B, respectively, then $S_A \cup S_B$ is called a standard generating set for $A \wr B$.

For $b, b' \in B$, and $F \subseteq B$ finite, denote by $\mathrm{TS}\left(b, b', F\right)$ the length of a shortest path in $\mathrm{Cay}(B, S_B)$ starting at b, finishing at b' and visiting all elements of F.

Lemma (Parry '92)

The word length of an element $g = (f, x) \in A \wr B$ with respect to $S_A \cup S_B$ is

$$|g|_{S_A \cup S_B} = \sum_{y \in \mathrm{supp}(f)} |f(y)|_{S_A} + \mathrm{TS}\left(e_B, x, \mathrm{supp}(f)\right).$$

Dead ends on lamplighter groups

- Cleary & Taback's Theorem generalizes to $A \wr F(S)$ where (A, S_A) has unbounded depth (in particular any finite group) and $F(S)$ is the free group on the set S.

- The argument strongly relies on the fact that the TSP has explicit solutions on a tree (i.e. the Cayley graph $\mathrm{Cay}(F(S), S)$)

Question: Does $A \wr B$ have unbounded depth for other base groups B? or for non-free generating sets of $F(S)$?

Fix a lamps group (A, S_A) with unbounded depth (e.g. any finite group).

Theorem (S. '22)

- *For every finitely generated B, there exists a finite genset S_B such that $(A \wr B, S_A \cup S_B)$ has unbounded depth.*
- *When B is abelian, any S_B works.*

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No!

Proposition (S. '22)

Consider $m, n \geq 2$ such that $m + n \geq 10$. Then

$$(A \wr (\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}), S_A \cup \{[1]_m, [1]_n\})$$

has uniformly bounded depth.