Poisson boundaries and entropy of random walks on groups

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World of GroupCraft V

Harmonic functions and the Poisson boundary

G a countable group, μ a probability measure on G.

A function $f : G \to \mathbb{R}$ is called μ-harmonic if $f(g) = \sum_{h \in G} f(gh)\mu(h)$ for all $g \in G$.

$$H^\infty(G,\mu) := \{f : G \to \mathbb{R} \mid f \text{ bounded and } \mu\text{-harmonic}\}$$

The Poisson boundary (B,ν) of (G,μ) is a probability space endowed with a G-action such that $\nu = \mu * \nu$ (i.e., ν is μ-stationary) and

$$H^\infty(G,\mu) \cong L^\infty(B,\nu).$$

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The Poisson boundary is **trivial** if and only if any bounded μ-harmonic function on G is **constant**.

Harmonic functions and the Poisson boundary

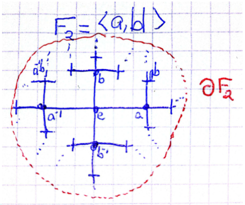
- The Poisson boundary goes back to the work of Furstenberg in the 60's, who used it to prove rigidity results for lattices in semisimple Lie groups.
- Poisson boundaries detect geometric properties, e.g.,
  - G is amenable if and only if there is some non-degenerate $\mu \in \mathrm{Prob}(G)$ with trivial PB [Rosenblatt '81, Kaimanovich-Vershik '83].
  - Suppose G finitely generated. Then G has a nilpotent subgroup of finite index if and only if *every* $\mu \in \mathrm{Prob}(G)$ has trivial PB [Blackwell '55, Choquet-Deny '60, Dynkin-Maljutov '61, Margulis '66, Lin-Zaidenberg '98, Jaworski '04, Frisch-Hartman-Tamuz-Vahidi Ferdowsi '19].
  - Suppose there is $\mu \in \mathrm{Prob}(G)$ that is *finitely supported* and has non-trivial PB. Then G has exponential growth.

Main question of today's talk:

How sensitive is the space of bounded μ-harmonic functions on G on small perturbations of μ?

Example 1: $G = F_2$ **the free group of rank** 2**.**

For any (non-degenerate) $\mu \in \mathrm{Prob}(F_2)$ there is a unique μ-stationary prob. measure ν on ∂F_2.



If $H(\mu) := -\sum_{g \in F_2} \mu(g) \log \mu(g) < \infty$, then $(\partial F_2, \nu)$ is the Poisson boundary of (G, μ). [Dynkin-Maljutov '61, Derriennic '75, Ancona '87, Kaimanovich '94, Chawla-Forghani-Frisch-Tiozzo '22]

(true more generally for hyperbolic groups G with Gromov boundary ∂G)

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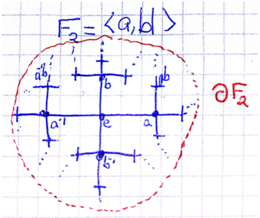

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⚠Not true in general if $H(\mu) = \infty$ [Chawla-Frisch '25]

Examples of Poisson boundaries

Example 2: $G = \mathbb{Z}/2\mathbb{Z} \wr B$ **the wreath product with** $\mathbb{Z}/2\mathbb{Z}$ **lamps and base group** B**.**

$$\mathbb{Z}/2\mathbb{Z} \wr B := \bigoplus_{B} \mathbb{Z}/2\mathbb{Z} \rtimes B.$$

where $\bigoplus_B \mathbb{Z}/2\mathbb{Z} = \{f : B \to \mathbb{Z}/2\mathbb{Z} \mid f \text{ with finite support}\}$ and B acts on $f \in \bigoplus_B \mathbb{Z}/2\mathbb{Z}$ by:

$$(b \cdot f)(x) = f(b^{-1}x), \; x, b \in B.$$

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Lamplighter interpretation

Multiplying an element $(f, x) \in \mathbb{Z}/2\mathbb{Z} \wr B$ on the right by elements of $\mathbb{Z}/2\mathbb{Z}$ changes the lamp configuration f at the current position x, while multiplying by elements of B changes said current position.

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Consider $\mu \in \mathrm{Prob}(\mathbb{Z}/2\mathbb{Z} \wr B)$ non-degenerate with $H(\mu) < \infty$.

Kaimanovich-Vershik '83, Erschler '04: the Poisson boundary of $(\mathbb{Z}/2\mathbb{Z} \wr B, \mu)$ is non-trivial if and only if the projection to B is transient.

Under general hypotheses on μ (e.g., a finite first moment), there is a μ-stationary prob. measure ν on $\prod_B \mathbb{Z}/2\mathbb{Z}$ such that $(\prod_B \mathbb{Z}/2\mathbb{Z}, \nu)$ is the Poisson boundary of $(\mathbb{Z}/2\mathbb{Z} \wr B, \mu)$. [Kaimanovich '00, Erschler '11, Lyons-Peres '21, Frisch-S. '24]

⚠In general there will be multiple μ-stationary measures (e.g., if B is amenable).

Consider G a countable group and $\mu \in \mathrm{Prob}(G)$ non-degenerate with $H(\mu) = -\sum_{g\in G} \mu(g) \log \mu(g) < \infty$.

The asymptotic entropy of (G, μ) is defined as

$$h(\mu) := \lim_{n\to\infty} \frac{H(\mu^{*n})}{n}.$$

The entropy criterion [Avez '72, Derriennic '81, Kaimanovich-Vershik '83]

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Entropy

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Question: Continuity of asymptotic entropy?

Fix G and consider $\mu, \{\mu_k\}_{k\geq 1} \in \mathrm{Prob}(G)$ with finite entropy such that:

- $\lim_{k\to\infty} \mu_k(g) = \mu(g)$ for each $g \in G$ and
- $\lim_{k\to\infty} H(\mu_k) = H(\mu)$.

Is it true that $\lim_{k\to\infty} h(\mu_k) = h(\mu)$?

Uniqueness of stationary measures on Poisson boundaries

Theorem [S. '25]

Let G be a countable group and consider non-degenerate prob. measures μ, $\{\mu_k\}_{k\geq 1}$ on G with finite entropy and such that $\mu_k \xrightarrow[k\to\infty]{} \mu$. Suppose that there is a Polish space X such that

- there are prob. measures ν (resp. ν_k) on X such that (X, ν) (resp. (X, ν_k)) is the Poisson boundary of (G, μ) (resp. (G, μ_k)).

If $\nu_k \xrightarrow[k\to\infty]{} \nu$ weakly, then $h(\mu_k) \xrightarrow[k\to\infty]{} h(\mu)$.

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If $\nu_k \xrightarrow[k\to\infty]{} \nu$ weakly, then $h(\mu_k) \xrightarrow[k\to\infty]{} h(\mu)$.

If X is furthermore compact and admits a unique μ-stationary prob. measure, then the weak convergence $\nu_k \xrightarrow[k\to\infty]{} \nu$ always holds.

Idea of the proof:

- Upper semicontinuity of asymptotic entropy follows from the subadditivity of the sequence $\{H(\mu^{*n})\}_{n\geq 1}$ [Amir-Angel-Virág '13]

$$\limsup_{k\to\infty} h(\mu_k) \leq h(\mu).$$

- Kaimanovich-Vershik '83:

$$h(\mu) = -\sum_{g\in G}\mu(g)\int_X \log\left(\frac{dg_*^{-1}\nu}{d\nu}(\xi)\right)d\nu(\xi) = \sum_{g\in G}\mu(g) I(g_*^{-1}\nu \mid \nu).$$

  Here $I(\cdot \mid \cdot)$ is the Kullback-Leibler distance.

- Consider $\{\nu_k\}_{k\geq 1}, \{\eta_k\}_{k\geq 1}, \nu, \eta \in \mathrm{Prob}(X)$ such that $\nu_k \xrightarrow[k\to\infty]{} \nu$ and $\eta_k \xrightarrow[k\to\infty]{} \eta$ weakly. Then

$$\liminf_{k\to\infty} I(\nu_k|\eta_k) \geq I(\nu|\eta).$$

Applications:

- Hyperbolic groups (Previously proved with different methods by Erschler-Kaimanovich ’13, Gouëzel-Mathéus-Maucourant ’18, Choi ’24).
- Acylindrically hyperbolic groups [Choi ’24] (e.g. mapping class groups, $\mathrm{Out}(F_n)$, etc...)
- $\mathrm{SL}_d(\mathbb{Z})$ for $d \geq 3$.
- Many groups acting on $\mathrm{CAT}(0)$-spaces.

Discontinuity of asymptotic entropy

Example: $\mathbb{Z}/2\mathbb{Z} \wr D_\infty$**, where** $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

- Gilch '08: any non-degenerate finitely supported prob. measure on D_∞ is recurrent.
- D_∞ has $\mathbb{Z}$ as a finite index subgroup (that admits finitely supported transient probability measures).
- There are $\mu, \{\mu_k\}_{k\geq 1} \in \mathrm{Prob}(\mathbb{Z}/2\mathbb{Z} \wr D_\infty)$ finitely supported such that $h(\mu) > 0$ and $h(\mu_k) = 0$ for all $k \geq 1$, with $\mu_k \xrightarrow[k\to\infty]{} \mu$.

Continuity of asymptotic entropy on wreath products

Theorem [S. '25]

Let B be a virtually nilpotent group of at least cubic growth (e.g. $B = \mathbb{Z}^d$, $d \geq 3$). For any non-degenerate probability measures $\mu, \{\mu_k\}_{k\geq 1}$ on $\mathbb{Z}/2\mathbb{Z} \wr B$ such that

- $\mu_k(g) \xrightarrow[k\to\infty]{} \mu(g)$ for all $g \in \mathbb{Z}/2\mathbb{Z} \wr B$ and
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Intuition: Entropy at time n on $\mathbb{Z}/2\mathbb{Z} \wr B \longleftrightarrow$ number of distinct points visited by the projection of the random walk to the base B up to time n.

Definition

Let μ be a prob. measure on a countable group G. Denote by $(X_n)_{n\geq 0}$ the μ-random walk on G. We define the escape probability

$$p_{\mathrm{esc}}(\mu) := \mathbb{P}(X_n \neq e_G \text{ for all } n \geq 1).$$

It holds almost surely that $\lim_{n\to\infty} \frac{\# |\{X_0, X_1, \ldots, X_n\}|}{n} = p_{\mathrm{esc}}(\mu)$.

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Theorem [S. '25]

Let G be a group of at least cubic growth (e.g. $G = \mathbb{Z}^d$, $d \geq 3$). For any non-degenerate probability measures $\mu, \{\mu_k\}_{k\geq 1}$ on G such that $\mu_k(g) \xrightarrow[k\to\infty]{} \mu(g)$ for all $g \in G$ we have $p_{\mathrm{esc}}(\mu_k) \xrightarrow[k\to\infty]{} p_{\mathrm{esc}}(\mu)$.

Uses (a uniform version of) a comparison lemma of heat kernels of symmetric and non-symmetric Markov operators due to Coulhon and Saloff-Coste (late 80's).