Bounded harmonic functions on groups acting on the circle

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The classical Poisson integral representation formula

Correspondence between bounded harmonic functions on the unit disk $\mathbb{D} \subseteq \mathbb{C}$ and bounded measurable functions on the circle $S^1 = \partial\mathbb{D}$.

Poisson kernel $P_r(\theta) := \frac{1-r^2}{1-2r\cos(\theta)+r^2}$, for $0 \le r < 1$ and $-\pi \le \theta < \pi$.

⟶ Let $F \in L^\infty(S^1)$ and define $u : \mathbb{D} \to \mathbb{R}$ by

$$u(re^{i\theta}) := \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it}) P_r(\theta - t)\, dt, \text{ for } 0 \le r < 1 \text{ and } -\pi \le \theta < \pi.$$

Then $u \in H^\infty(\mathbb{D})$ and its extension to S^1 coincides with F.

⟶ Any $u \in H^\infty(\mathbb{D})$ admits such a representation.

Group theory in the background: one can rewrite for each $z \in \mathbb{D}$

$$u(z) = \int_{S^1} F(\xi)\, dg_*\mathrm{Leb}(\xi),$$

where $g \in \mathrm{PSL}(2,\mathbb{R})$ satisfies $g(0) = z$. (actually $g \in \mathrm{PSL}(2,\mathbb{R})/PSO(2)$)

Harmonic functions and the Poisson boundary

G a countable group, μ a probability measure on G.

A function $f : G \to \mathbb{R}$ is called μ-harmonic if $f(g) = \sum_{h \in G} f(gh)\mu(h)$ for all $g \in G$.

$$H^\infty(G,\mu) := \{f : G \to \mathbb{R} \mid f \text{ bounded and } \mu\text{-harmonic}\}$$

The Poisson boundary (B,ν) of (G,μ) is a probability space endowed with a G-action such that $\nu = \mu * \nu$ (i.e., ν is μ-stationary) and

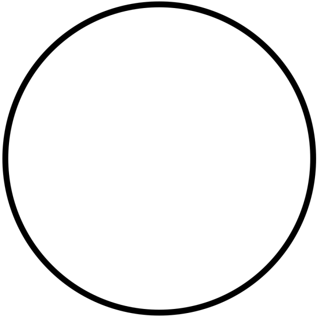
$$H^\infty(G,\mu) \cong L^\infty(B,\nu).$$

(uniquely defined up to a G-equivariant measurable iso.; satisfies a universal property)

The circle

Rotations $\curvearrowright$

Lattices in $PSL(2,\mathbb{R})$ (e.g. surface groups $\pi_1(\Sigma_g)$) $\curvearrowright$



$\curvearrowleft$ Thompson's group T (i.e., piecewise linear dyadic orientation-preserving homeos. of S^1)

Source: https://www.kidsmathgamesonline.com/facts/geometry/circles.html

Theorem [Deroin-Kleptsyn-Navas '07]

Let $G \curvearrowright S^1$ by orientation-preserving homeos. with no invariant probability measure on S^1 and let $\mu \in \mathrm{Prob}(G)$ be non-degenerate. Suppose that $G \curvearrowright S^1$ is proximal. Then there is a **unique** μ-stationary probability measure on S^1.

Theorem [Gilabert Vio - Kravaris - S. '25]

Let μ be a probability measure with finite entropy on a countable group G of orientation-preserving homeomorphisms of the circle acting proximally, minimally and topologically nonfreely on S^1. Then the circle S^1 endowed with its unique μ-stationary probability measure **is not** the Poisson boundary of (G, μ).

- ⟶ This contrasts with the case of lattices in $\mathrm{PSL}_2(\mathbb{R})$, in which case the circle **is** the Poisson boundary.
- ⟶ Applies in particular when G is **Thompson's group** T and μ is finitely supported. This answers a question asked by B. Deroin and A. Navas [Proceedings of the ICM, 2018].
- ⟶ Shows that a particular strategy for proving the amenability of Thompson's group F is (basically?) hopeless.