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Asymptotic properties and Poisson boundaries of wreath product-like groups

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Abstract

In this thesis, we investigate combinatorial, geometric, and probabilistic properties of wreath products and other group extensions. The work is divided into the following two parts.

Non-extendable geodesics in Cayley graphs. We study the property of having *unbounded depth* in Cayley graphs of wreath products. That is, whether there exist elements at arbitrarily large distance from other elements of larger word length. We prove that for any finite group A and any finitely generated group B, the wreath product $A \wr B$ admits a standard generating set with unbounded depth. If B is abelian, then the above is true for every standard generating set. This generalizes the case $B = \mathbb{Z}$, due to Cleary and Taback. When $B = H * K$ for two finite groups H and K, we characterize which standard generators of $A \wr B$ have unbounded depth in terms of a geometrical constant related to the Cayley graphs of H and K.

Random walks and Poisson boundaries of groups. First, we study random walks on the lampshuffler group $\mathrm{FSym}(H) \rtimes H$, where H is a finitely generated group and $\mathrm{FSym}(H)$ is the group of finitary permutations of H. We show that for any step distribution μ with a finite first moment that induces a transient random walk on H, the permutation coordinate of the random walk almost surely stabilizes pointwise to a limit function. Our main result states that for $H = \mathbb{Z}$, the Poisson boundary of the random walk $(\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}, \mu)$ is equal to the space of limit functions endowed with the hitting measure. Our result provides new examples of completely described non-trivial Poisson boundaries of elementary amenable groups.

Next, in collaboration with Joshua Frisch, we completely describe the Poisson boundary of the wreath product $A \wr B$ of countable groups A and B, for all probability measures μ with finite entropy and such that the lamp configurations stabilize almost surely along sample paths. If in addition the projection of μ to B is Liouville, we prove that the Poisson boundary of $(A \wr B, \mu)$ coincides with the space of limit lamp configurations, endowed with the corresponding hitting measure. This improves earlier results by Lyons-Peres and, in particular, we answer an open question asked by Kaimanovich and Lyons-Peres for $B = \mathbb{Z}^d$, $d \geq 3$, and measures μ with a finite first moment.

Keywords : wreath products, random walks, Poisson boundary, dead ends

Résumé

Dans cette thèse, nous étudions des propriétés combinatoires, géométriques, et probabilistes des produits en couronne ainsi que d'autres extensions de groupes. Ce travail est divisé en deux parties.

Géodésiques non extensibles dans des graphes de Cayley. Nous étudions la propriété d'avoir *profondeur non-bornée* dans les graphes de Cayley des produits en couronne. En d'autres termes, nous cherchons à savoir s'il existent des éléments situés à une distance arbitrairement grande d'autres éléments ayant une longueur de mot plus grande. Nous prouvons que pour tout groupe fini A et tout groupe de type fini B, le produit en couronne $A \wr B$ admet un ensemble de générateurs standard avec une profondeur non bornée. Si B est abélien, alors ce qui précède est vrai pour tout ensemble générateur standard. Ceci généralise le cas $B = \mathbb{Z}$, dû à Cleary et Taback. Lorsque $B = H * K$ pour deux groupes finis H et K, nous caractérisons quels générateurs standards de $A \wr B$ ont une profondeur non bornée en termes d'une constante géométrique liée aux graphes de Cayley de H et K.

Marches aléatoires et bords de Poisson des groupes. D'abord, nous étudions des marches aléatoires sur le groupe $\mathrm{FSym}(H) \rtimes H$, où H est un groupe de type fini et $\mathrm{FSym}(H)$ est le groupe des permutations de support fini de H. Nous montrons que pour toute distribution μ des incréments avec un premier moment fini induisant une marche aléatoire transiente sur H, la coordonnée de permutation de la marche aléatoire se stabilise presque sûrement. Notre résultat principal affirme que le bord de Poisson de la marche aléatoire $(\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}, \mu)$ est égal à l'espace des fonctions limites doté de la mesure harmonique correspondante. Cela fournit de nouveaux exemples de bords de Poisson non-triviaux complètement décrits pour un groupe élémentairement moyennable.

Ensuite, en collaboration avec Joshua Frisch, nous décrivons complètement le bord de Poisson du produit en couronne $A \wr B$ des groupes dénombrables A et B, pour toutes les mesures de probabilité μ d'entropie finie et telles que les configurations de lampe se stabilisent presque sûrement. Si en plus la projection de μ sur B a la propriété de Liouville, le bord de Poisson de $(A \wr B, \mu)$ est égal à l'espace de configurations de lampes limites, doté de la mesure harmonique. Cela généralise des résultats précédents de Lyons-Peres pour $d \geq 3$ et, en particulier, nous répondons à une question ouverte posée par Kaimanovich et Lyons-Peres pour $B = \mathbb{Z}^d$, $d \geq 3$, et des mesures μ avec un premier moment fini.

Mots clés : produits en couronne, marches aléatoires, bord de Poisson, impasses, culs-de-sac

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Introduction

The work developed in this thesis is in the area of *Geometric Group Theory*. This field has as a general philosophy to look at groups as geometric objects and to understand their properties by using tools from multiple areas such as geometry, topology, combinatorics, probability, and ergodic theory. In this thesis we will concentrate on the case of infinite, finitely generated groups, and the work is divided into two parts: the first one is about asymptotic metric properties of Cayley graphs, and the second one is about random walks on groups and the identification of Poisson boundaries. In both cases, the groups we consider are wreath products and other group extensions, which we discuss in more detail below.

A natural family of metric spaces associated with a finitely generated group are its *Cayley graphs*. Given a finite generating set S of G, the Cayley graph $\mathrm{Cay}(G, S)$ is the graph that has G as a vertex set and where edges connect elements g and gs, for $g \in G$ and $s \in S \cup S^{-1}$. Although different choices of generating set lead to different Cayley graphs, they are all *quasi-isometric* to each other: the metrics on the group G induced by the graph structure of two distinct Cayley graphs are the same up to a composition with a linear transformation. Similarly, if G has a subgroup of finite index H, the metrics of G and H associated with arbitrary finite generating sets will be quasi-isometric to each other.

For certain groups, it is relatively simple to visualize the large-scale geometry of their Cayley graphs. This is the case for free abelian groups and hyperbolic groups, which resemble Euclidean and hyperbolic spaces, respectively. However, for some groups, this task becomes more challenging because their Cayley graphs may exhibit unexpected geometric behavior. This is the case for wreath products of groups, which are the main family of groups that we study in this thesis.

Wreath products $A \wr B := \bigoplus_B A \rtimes B$ of groups A and B have been well studied in the past decades, and they provide examples and counterexamples of diverse geometric properties of groups. Let us illustrate this with the following example: One can associate with each finitely generated group its *growth type*, which measures the asymptotic behavior of the volume growth of balls in its Cayley graphs. Every finitely generated nilpotent group has polynomial growth type [Wolf, 1968], and their growth functions are equivalent to a polynomial n^d, where $d \in \mathbb{N}$ is an integer determined by the lower central series of G [Bass, 1972; Guivarch, 1973]. Gromov's Theorem of polynomial growth states that every finitely generated group of polynomial growth has a nilpotent subgroup of finite index [Gromov, 1981b]. By using these results, it is possible to prove that the algebraic property of having a nilpotent (resp. abelian) subgroup of finite index is geometric, in the sense that any group that is quasi-isometric to a finitely generated group with a nilpotent (resp. abelian) finite index subgroup must also have this property. In contrast, wreath products show that the property of having a solvable subgroup of finite index is not geometric [Dyubina, 2000]. More precisely, if we denote by A_5 the alternating group in 5 elements, then the wreath products $(A_5 \times \mathbb{Z}) \wr \mathbb{Z}$ and $\mathbb{Z} \wr \mathbb{Z}$ are quasi-isometric, but the first one is not virtually solvable whereas the second one is 2-step solvable. Using the same example, it can also be seen that the property of having a torsion-free finite index subgroup is not geometric.

Wreath products of the form $F \wr \mathbb{Z}$, for F a finite non-trivial group, were classified up to quasi-isometries by [EskinFisherWhyte, 2012; EskinFisherWhyte, 2013] by introducing the technique of "coarse differentiation". Following this, [Dymarz, 2010] showed that groups $F \wr \mathbb{Z}$ as above provide examples of groups that are quasi-isometric but not bi-Lipschitz equivalent. Such an example can only occur within amenable groups, since the notions of quasi-isometry and bi-Lipschitz equivalence coincide for non-amenable groups [Nekrashevych, 1998; Whyte, 1999]. Recently, a classification up to quasi-isometry of groups $F \wr H$ with F finite and H one-ended was obtained in [GenevoisTessera, 2021], through different techniques to those of Eskin-Fisher-Whyte, and which rely on quasi-median geometry.

The geometric properties of the Cayley graph of $A \wr B$ with respect to standard generating sets are directly related to a combinatorial problem in the associated Cayley graph of B. Namely, the word length of an element of $A \wr B$ can be expressed in terms of the word length of A, and the solutions to the *Traveling Salesperson Problem* (abbreviated as TSP) in the Cayley graph of B. This connection was noted in [Parry, 1992], who showed that wreath products of finite groups with free groups have algebraic growth series that are not rational. Since then, this description of the word length has been used on various occasions to study the geometric properties of wreath products.

Part 1: Unbounded depth on wreath products

In the first part of this thesis, we use the relation between the word length on $A \wr B$ and the TSP to study the *depth properties* of the Cayley graph of $A \wr B$. Given a Cayley graph $\mathrm{Cay}(G, S)$ of a group G, it is natural to ask whether every element $g \in G$ is connected to another element of strictly larger word length. Elements that do not satisfy this property are called *dead ends*. The existence of dead ends is sensitive to the local geometry of the Cayley graph, and in any infinite finitely generated group, it is possible to find some generating set with dead ends [Šunić, 2008]. In particular, the existence of dead ends is not invariant under changing the generating set S by another one. A large-scale version of this problem is to ask whether the distance between a dead end g and any element with word length $|g|_S + 1$ can be arbitrarily large. In such a case, we say that the group has *unbounded depth of dead ends*. This property is invariant under $(1, C)$-quasi-isometries, which corresponds to changing the metric by a bounded additive constant.

The unbounded depth property gives structural information about the stratification of G by spheres $S_n = \{g \in G \mid |g|_S = n\}$, $n \geq 1$. In the case of abelian groups and hyperbolic groups, there are no generating sets with unbounded depth [Bogopolski, 1997; Lehnert, 2009]. More generally, this holds whenever there is a regular language composed of geodesic words that evaluate to every group element [Warshall, 2010]. In this sense, having unbounded depth is an obstruction to some of the computability properties of the group. This is further explored in [BagnoudBodart, 2022], where it is shown that having unbounded depth implies the irrationality of the $\mathbb{N}G$-growth series of the group.

The first example of a finitely generated group with unbounded depth was $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ [Cleary-Taback, 2005a]. Their proof relies on the fact that in this one-dimensional base group there are explicit solutions to the TSP, and hence the word metric on the wreath product has an explicit expression that allows for direct computations. It is not clear that the property of having unbounded depth is preserved when considering more general wreath products, since now one cannot expect to obtain explicit solutions to the TSP. Indeed, this problem is known to be NP-complete if $B = \mathbb{Z}^2$.

We studied this more general problem in the case of lamp groups with unbounded depth, and proved that it is always possible to find standard generators of the corresponding wreath product that have unbounded depth.

Theorem (= Theorem 2.4.8)**.** *Let (A, S_A) have unbounded depth and B be any finitely generated group. Then there exists a finite generating set S_B of B for which $(A \wr B, S_A \cup S_B)$ has unbounded depth.*

If we additionally suppose that the base group B is abelian, the above holds for every choice of S_B (Proposition 2.4.15). We remark that the hypothesis that (A, S_A) has unbounded depth is necessary, since otherwise at any element of $A \wr B$ one could multiply by elements of A in order to increase the word length, and ignore the wreath product structure.

The proof of this result relies on the fact that we do not need to understand the exact solutions to the TSP on the Cayley graph of B, but rather only their length. In the case of abelian groups, we use the property that there are finite subgraphs of their Cayley graphs that contain arbitrarily large balls, in which the solution to the TSP corresponds to a path that is close to being Hamiltonian (i.e. that only visits a bounded number of vertices more than once).

Additionally, we also prove that one cannot expect to obtain unbounded depth for every choice of standard generating set in every wreath product $A \wr B$ with B non-abelian.

Theorem (= Theorem 2.5.3)**.** *Let (A, S_A) have unbounded depth, and consider two finite groups H and K with generating sets S_H and S_K, respectively. Then $(A \wr (H * K), S_A \cup S_H \cup S_K)$ does not have unbounded depth if and only if $\mathscr{H}(H, S_H) + \mathscr{H}(K, S_K) \geq 1$.*

Here the value of $\mathscr{H}(G, S_G)$, for G a finite group, quantifies of how much shorter a minimal spanning cycle of $\mathrm{Cay}(G, S_G)$ is in comparison with minimal spanning paths from e_G to a non-identity element (Definition 2.5.2). The condition $\mathscr{H}(H, S_H) + \mathscr{H}(K, S_K) \geq 1$ above holds in particular whenever $H = \mathbb{Z}/n\mathbb{Z}$ and $K = \mathbb{Z}/m\mathbb{Z}$ with $m, n \geq 2$ and $m + n \geq 10$.

Part 2: Random walks and the identification of Poisson boundaries

In the second part of this thesis, we study the asymptotic behavior of random walks on groups. Given a countable group G and a probability measure μ on G, the *μ-random walk on G* is the Markov chain $\{w_n\}_{n\geq 0}$ with state space G and transition probabilities $p(x,y)=\mu(y^{-1}x)$, for $x,y\in G$. We assume that the initial distribution is concentrated in the identity of G, so that $w_0=e_G$. In the case of finitely generated groups, a natural family of probability measures are those distributed uniformly on a symmetric finite generating set. The corresponding random walks are referred to as *simple random walks* on G.

In this thesis, we study the identification problem for the Poisson boundary of random walks on groups, which we discuss below.

The *Poisson boundary* of the μ-random walk on G is a probability space that encodes the asymptotic behavior of sample paths, and it admits several equivalent definitions (see Section 3.2). The Poisson boundary has been used on multiple occasions to prove results on groups that are not related to random walks, such as in Furstenberg's approach to superrigidity theorems [Furstenberg, 1963; Furstenberg, 1971], in Bader-Shalom's Normal Subgroup Theorem [Bader-Shalom, 2006], and in proving near-optimal volume growth estimates for Grigorchuk's group of intermediate growth [ErschlerZheng, 2020].

The amenability of a group is completely determined by the Poisson boundary of the random walks defined by non-degenerate measures (i.e. such that their support generates the group as a semigroup). More precisely, in every non-amenable group, every random walk with step increments given by a non-degenerate measure has a non-trivial Poisson boundary [Azencott, 1970, Proposition II.1] (see also the end of Section 9 in [Furstenberg, 1973]). Conversely, every amenable group admits a random walk driven by a probability measure supported on the whole group that has a trivial Poisson boundary [KaimanovichVershik, 1983; Rosenblatt, 1981]. Because of this result, amenable groups can be divided into two classes. The first one is composed of groups in which *every* non-degenerate random walk has a trivial Poisson boundary. These groups are usually called "Choquet-Deny groups", and in the countable case, they are precisely the family of groups that do not admit a non-trivial quotient where every non-trivial element has an infinite conjugacy class [FrischHartmanTamuzVahidi Ferdowsi, 2019] (see also the references in [FrischHartmanTamuzVahidi Ferdowsi, 2019] for the previous work on this problem). In particular, this equivalence says that a finitely generated group is Choquet-Deny if and only if it is virtually nilpotent. The second class of amenable groups are those that admit non-degenerate measures with a non-trivial Poisson boundary. As proved by Kaimanovich-Vershik, wreath products of the form $\mathbb{Z}/2\mathbb{Z}\wr\mathbb{Z}^d$, $d\geq 1$, belong to this family [KaimanovichVershik, 1983]. Moreover, the case $d\geq 3$ gave the first examples of amenable groups that admit simple random walks with a non-trivial Poisson boundary. Examples of non-symmetric random walks on amenable groups with a non-trivial Poisson boundary had been described in [Furstenberg, 1973, last paragraph of Section 9] for the Baumslag-Solitar group $\mathrm{BS}(1,2)$.

A key quantity in the study of Poisson boundaries is *entropy*. The entropy of the measure μ on G is $H(\mu) := -\sum_{g\in G}\mu(g)\log(\mu(g))$. For probability measures with $H(\mu)<\infty$, the *entropy criterion* of Avez [Avez, 1972], Derrienic [Derriennic, 1980] and Kaimanovich-Vershik

[KaimanovichVershik, 1983] states that the triviality of the Poisson boundary is equivalent to the vanishing of the *asymptotic entropy* $h(\mu) \coloneqq \lim_{n\to\infty} H(\mu^{*n})/n$. This criterion was extended to a *conditional entropy criterion* by Kaimanovich [Kaimanovich, 2000, Theorem 4.6], that establishes whether a candidate for the Poisson boundary is equal to it.

Recall that a group is called locally finite if every finitely generated group is finite, and that a group extension of a locally finite group by a cyclic group $\mathbb{Z}$ is called a *locally-finite-by-$\mathbb{Z}$* group. A particular family of examples of the latter are wreath products $F \wr \mathbb{Z}$ for F a finite non-trivial group. Locally-finite-by-$\mathbb{Z}$ groups can have a large range of different geometric properties. For instance, they can have arbitrarily fast-growing Følner functions (see [Erschler, 2003], [Gromov, 2008, Section 8.2] and [OlshanskiiOsin, 2013, Corollary 1.5]). In [BrieusselZheng, 2021, Theorem 1.1], it is proved that it is possible to find cyclic extensions of locally finite groups with a prescribed speed function, decay of return probability, entropy function, ℓ^p-isoperimetric profile, and L_p-compression function. In particular, the authors prove that any sufficiently regular function that grows at least exponentially can be realized as the Følner function of a locally-finite-by-$\mathbb{Z}$ group, which admits a simple random walk with a trivial Poisson boundary [BrieusselZheng, 2021, Corollary 4.7].

We studied random walks on the locally-finite-by-$\mathbb{Z}$ group $\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}$. Here $\mathrm{FSym}(\mathbb{Z})$ is the group of finitely supported permutations of $\mathbb{Z}$. We refer to this group as a "lampshuffler" by making an analogy with lamplighter groups, where instead of turning lamps on and off, they are being permuted. Random walks on $\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}$ have been studied in [Yadin, 2009], where it is proved that the drift function of the simple random walk for its standard generating is asymptotically equivalent to $n^{3/4}$. This implies that the simple random walk on the lampshuffler group has a trivial Poisson boundary. In [ErschlerZheng, 2020, Corollary 1.4] it is shown that the Følner function of $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$, $d \geq 1$, is asymptotically equivalent to n^{n^d}, and the return probability $\mu^{2n}(e)$ of the simple random walk is shown to be asymptotically $\exp\left(-n^{\frac{d}{d+2}} \log^{\frac{2}{d+2}} n\right)$.

Every random walk on $\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}$ naturally induces a random walk on $\mathbb{Z}$ via the canonical projection. If this random walk is transient and the original measure has a finite first moment, it can be shown that the first coordinate of the random walk associated with the permutation group $\mathrm{FSym}(\mathbb{Z})$ asymptotically stabilizes to a limit function. We proved that in this situation, this convergence fully describes the associated Poisson boundary.

Theorem (= Theorem 4.1.2)**.** *Consider a probability measure μ on $\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}$. Suppose that μ has a finite first moment, and that its projection to $\mathbb{Z}$ induces a transient random walk. Then the Poisson boundary of μ is completely described by the space of limit functions, endowed with the hitting measure.*

The first description of a non-trivial Poisson boundary of an amenable group was given by Kaimanovich-Vershik [KaimanovichVershik, 1983], who proved that simple random walks on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$, $d \geq 3$, have a non-trivial Poisson boundary. Since then, the identification problem of the Poisson boundary of wreath products $A \wr B$ has been studied under various moment conditions on the probability measure μ, in the cases of A finite or finitely generated, and where $B = \mathbb{Z}^d$ [Erschler, 2011; JamesPeres, 1996; Kaimanovich, 2001; LyonsPeres, 2021a], B a non-abelian free group [KarlssonWoess, 2007], and B a non-elementary hyperbolic group [Sava, 2010].

We can associate with every group element in the wreath product $A \wr B$ a *lamp configuration* in $\bigoplus_B A$. Given a probability measure μ on $A \wr B$ with a finite support and that induces a transient random walk on B, the lamp configuration will stabilize along sample paths to a limit lamp configuration. Kaimanovich and Vershik asked whether, under the previous hypotheses on μ, the space of limit lamp configurations endowed with the corresponding hitting measure completely describes the Poisson boundary of the μ-random walk on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$ [KaimanovichVershik, 1983]. This question was answered positively by Erschler for $d \geq 5$ and probability measures μ with a finite third moment [Erschler, 2011], and by Lyons-Peres for $d \geq 3$ and μ with a finite second moment [LyonsPeres, 2021a].

More generally, the stabilization of lamp configurations also occurs for infinitely supported probability measures that have a finite first moment and that induce a transient random walk on $\mathbb{Z}^d$. It was asked by Kaimanovich [Kaimanovich, 2001, Example 3.6.7] and by Lyons-Peres [LyonsPeres, 2021a, Section 5] whether, for this larger class of probability measures, the space of limit lamp configurations completely describes the Poisson boundary of $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$. In collaboration with Joshua Frisch, we give a positive answer to this question.

Theorem (= Theorem 5.1.6)**.** *Let A, B be finitely generated groups and μ a non-degenerate probability measure on $A \wr B$. Assume that μ has a finite first moment and that it induces a transient random walk on B. Then the space of limit lamp configurations A^B endowed with the corresponding hitting measure is equal to the Poisson boundary of $(A \wr B, \mu)$.*

Moreover, we prove the following result, which does not have any hypotheses on the moments of the measure μ.

Theorem (= Theorem 5.1.3)**.** *Consider a non-trivial countable group A. Let μ be a probability measure on $A \wr \mathbb{Z}^d$, $d \geq 1$, with finite entropy and such that lamp configurations stabilize almost surely. Then the Poisson boundary of $(A \wr \mathbb{Z}^d, \mu)$ is completely described by the space $A^{\mathbb{Z}^d}$, endowed with the corresponding hitting measure.*

For this result to hold, it is necessary that we have the hypothesis of stabilization of lamp configurations along sample paths. This is not always the case, and there are examples where this does not happen for measures μ with a finite $(1-\varepsilon)$-moment, for any $\varepsilon > 0$ [Kaimanovich, 1983, Proposition 1.1] (see also [Erschler, 2011, Section 6] and the last paragraph of Section 5 in [LyonsPeres, 2021a]). Nonetheless, it is true that any non-degenerate random walk on $A \wr B$ with finite entropy and a transient projection to B has a non-trivial Poisson boundary [Erschler, 2004b, Theorem 3.1]. In this case, the description of the Poisson boundary seems more challenging, as there are not even any known candidates.

The interest in considering measures with a possibly infinite first moment comes from the fact that such measures are an important part of the theory of Poisson boundaries. The correspondence between amenability and the existence of fully supported measures with a trivial Poisson boundary is no longer true if we restrict ourselves to finite first moment measures. Indeed, the group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^3$ is amenable, but every non-degenerate measure with finite first moment (and more generally, with finite entropy) has a non-trivial Poisson boundary [Erschler, 2004b, Theorem 3.1]. Additionally, it follows from the entropy criterion that every measure with a finite first moment on a group of intermediate growth has a trivial Poisson boundary. Nonetheless, it is a particular consequence of the classification of Choquet-Deny groups of [FrischHartmanTamuzVahidi Ferdowsi, 2019] that every finitely generated group of intermediate growth admits a probability measure with a non-trivial Poisson boundary. This result does not guarantee any control over the tail decay of the measure. Probability measures on groups of intermediate growth with a finite α-moment, for $0 < \alpha < 1$, are constructed in [Erschler, 2004a; ErschlerZheng, 2020] and they provide volume growth estimates for the groups. In particular, [ErschlerZheng, 2020] used this to give near-optimal lower bounds for the volume growth of Grigorchuk's group. Results describing a non-trivial Poisson boundary usually assume the finiteness of some moment of μ. The only known results that describe a non-trivial Poisson boundary for all measures with finite entropy are [ForghaniTiozzo, 2019, Theorem 1.2] for the free semigroup and [ChawlaForghaniFrischTiozzo, 2022] for hyperbolic groups and more generally acylindrically hyperbolic groups. In contrast, there are very few results that identify a non-trivial Poisson boundary for infinite entropy measures on groups. One known exception is [ErschlerKaimanovich, 2023, Theorem A], which describes the Poisson boundary of ICC groups for a class of measures from the construction of [FrischHartmanTamuzVahidi Ferdowsi, 2019]. Additionally, [ForghaniKaimanovich, 2015] describes the Poisson boundary of the free semigroup for measures with a finite first logarithmic moment without assuming finite entropy (see also [ForghaniTiozzo, 2019, Theorem 1.2] and [Forghani, 2015, Theorem 3.6.3]). It is important to remark that even in the case of a non-abelian free semigroup, it is an open problem whether the space of infinite word endowed with the corresponding hitting measure completely describes the Poisson boundary for all measures with finite entropy.

Organization

Chapter 1 provides the necessary background on Geometric Group Theory for this thesis. We also discuss various characterizations of amenability and recall some of the geometric properties of wreath products that have been studied in the past decades. The rest of the manuscript is divided into two parts. The first part concerns the geometry of Cayley graphs of wreath products, and consists of Chapter 2 whichi is a reproduction of the article [Silva, 2023a]. The second part of this manuscript is about random walks on groups. In Chapter 3 we recall basic properties of random walks on groups, give several equivalent definitions of the Poisson boundary, discuss the entropy criteria associated with the non-triviality and identification of Poisson boundaries, and mention known results about the identification of Poisson boundaries of groups. The next two chapters are reproductions of articles: Chapter 4 corresponds to the preprint [Silva, 2023b], and Chapter 5 corresponds to the preprint [FrischSilva, 2023], written in collaboration with Joshua Frisch.

Introduction en français

Les travaux développés dans cette thèse appartiennent au domaine de la théorie géométrique des groupes. Ce domaine a pour philosophie générale de considérer les groupes comme des objets géométriques, et de comprendre leurs propriétés en utilisant des outils provenant de plusieurs domaines tels que la géométrie, la topologie, la combinatoire, les probabilités et la théorie ergodique. Dans cette thèse, nous nous concentrons sur le cas des groupes infinis et de type fini, et le travail est divisé en deux parties : la première concerne les propriétés géométriques asymptotiques des graphes de Cayley, et la deuxième concerne les marches aléatoires sur les groupes et l'identification des bords de Poisson. Dans les deux cas, les groupes que nous considérons sont des produits en couronne et d'autres extensions de groupes, que nous discutons plus en détail ci-dessous.

Une famille naturelle d'espaces métriques associés à un groupe de type fini est celle des *graphes de Cayley.* Étant donné un ensemble fini de générateurs S de G, le graphe de Cayley $\mathrm{Cay}(G, S)$ est le graphe dont l'ensemble des sommets est G et dont les arêtes relient les éléments g et gs, pour $g \in G$ et $s \in S \cup S^{-1}$. Bien que des choix différents de l'ensemble générateur S conduisent à des graphes de Cayley différents, ils sont tous *quasi-isométriques* les uns aux autres : les métriques sur le groupe G induites par la structure de graphe des deux graphes de Cayley distincts ne diffèrent que par une composition avec une transformation linéaire. De même, si G possède un sous-groupe d'indice fini H, les métriques de G et H associées à des ensembles générateurs finis arbitraires seront quasi-isométriques l'une à l'autre.

Pour certains groupes, il est relativement simple de visualiser la géométrie à grande échelle de leurs graphes de Cayley. C'est le cas des groupes abéliens libres et des groupes hyperboliques, qui ressemblent respectivement aux espaces euclidiens et hyperboliques. Cependant, pour certains groupes, cette tâche devient plus difficile car leurs graphes de Cayley peuvent présenter un comportement géométrique inattendu. C'est le cas des produits en couronne de groupes, qui constituent la principale famille de groupes que nous étudions dans cette thèse.

Les produits en couronne $A \wr B := \bigoplus_B A \rtimes B$ des groupes A et B ont été étudiés extensivement au cours des dernières décennies, et ils fournissent des exemples et des contre-exemples de diverses propriétés géométriques des groupes. Illustrons ceci par l'exemple suivant : On peut associer à chaque groupe de type fini son *type de croissance*, qui mesure le comportement asymptotique de la croissance du volume des boules dans ses graphes de Cayley. Tout groupe nilpotent de type fini a un type de croissance polynomial [Wolf, 1968], et leurs fonctions de croissance sont équivalentes à un polynôme n^d, où $d \in \mathbb{N}$ est un entier déterminé par la série centrale inférieure de G [Bass, 1972; Guivarch, 1973]. Le théorème de Gromov sur la croissance polynomiale affirme que tout groupe de type fini à croissance polynomiale possède un sous-groupe nilpotent d'indice fini [Gromov, 1981b]. En utilisant ces résultats, il est possible de prouver que la propriété algébrique d'avoir un sous-groupe nilpotent (resp. abélien) d'indice fini est géométrique, au sens où tout groupe quasi-isométrique à un groupe de type fini avec un sous-groupe nilpotent (resp. abélien) d'indice fini doit également avoir cette propriété. Par contre, les produits en couronne montrent que la propriété d'avoir un sous-groupe résoluble d'indice fini n'est pas géométrique [Dyubina, 2000]. Plus précisément, si l'on désigne par A_5 le groupe alterné de degré 5, alors les produits en couronne $(A_5 \times \mathbb{Z}) \wr \mathbb{Z}$ et $\mathbb{Z} \wr \mathbb{Z}$ sont quasi-isométriques, mais le premier n'est pas virtuellement résoluble alors que le deuxième est résoluble de classe 2. En utilisant le même exemple, on peut voir que la propriété d'avoir un sous-groupe d'indice fini sans torsion n'est pas géométrique.

Les produits en couronne de la forme $F \wr \mathbb{Z}$, pour F un groupe fini non trivial, ont été classifiés a quasi-isométrie près par [EskinFisherWhyte, 2012; EskinFisherWhyte, 2013] en introduisant une technique appelé "coarse differentiation" en anglais, ce qui pourrait être traduit comme "différentiation à grande échelle" en français. Par la suite, [Dymarz, 2010] a montré que les groupes $F \wr \mathbb{Z}$ comme ci-dessus fournissent des exemples de groupes qui sont quasi-isométriques mais non bi-Lipschitz équivalents. Un tel exemple ne peut se produire qu'avec des groupes moyennables, puisque les notions de quasi-isométrie et d'équivalence bi-Lipschitz coïncident pour les groupes non-moyennables [Nekrashevych, 1998; Whyte, 1999]. Récemment, une classification à quasi-isométrie près des groupes $F \wr H$ avec F fini et H avec un bout a été obtenue dans [GenevoisTessera, 2021], en utilisant des techniques différentes de celles d'Eskin-Fisher-Whyte et qui s'appuient sur la géométrie des espaces quasi-médians.

Les propriétés géométriques du graphe de Cayley de $A \wr B$ par rapport aux ensembles générateurs standard sont directement liées à un problème combinatoire dans le graphe de Cayley associé de B. En particulier, la longueur des mots d'un élément de $A \wr B$ peut être exprimée en termes de la longueur des mots de A, et les solutions au *problème du voyageur de commerce* (qui est abrégé en TSP en raison de son nom anglais) dans le graphe de Cayley de B. Ce lien a été remarqué par [Parry, 1992], qui a montré que les produits en couronne des groupes finis avec des groupes libres ont des séries de croissance algébriques qui ne sont pas rationnelles. Depuis, cette description de la longueur des mots a été utilisée à plusieurs reprises pour étudier les propriétés géométriques des produits en couronne.

Partie 1 : Profondeur non-bornée dans les produits en couronne

Dans la première partie de cette thèse, nous utilisons la relation entre la longueur des mots sur $A \wr B$ et le TSP pour étudier les *propriétés de profondeur* du graphe de Cayley. Étant donné un graphe de Cayley $\mathrm{Cay}(G, S)$ d'un groupe G, il est naturel de se demander si chaque élément $g \in G$ est connecté à un autre élément de longueur de mot strictement plus grande. Les éléments qui ne satisfont pas à cette propriété sont appelés des "culs-de sac" ou "impasses". L'existence de culs-de-sac est sensible à la géométrie locale du graphe de Cayley, et dans tout groupe infini de type fini il est possible de trouver un ensemble de générateurs avec des culs-de-sac [Šunić, 2008]. En particulier, l'existence de culs-de-sac n'est pas invariante par rapport à un changement de l'ensemble de générateurs S par un autre. Une version à grande échelle de ce problème consiste à se demander si la distance entre un cul-de-sac g et tout élément ayant une longueur de mot $|g|_S + 1$ peut être arbitrairement grande. Dans ce cas, nous disons que le groupe a une profondeur non-bornée de culs-de-sac. Cette propriété est invariante sous $(1, C)$-quasi-isométries, ce qui correspond à changer la métrique par une constante additive bornée.

La propriété de profondeur non-bornée donne des informations structurelles sur la stratification de G par des sphères $S_n = \{g \in G \mid |g|_S = n\}$, $n \geq 1$. Dans le cas des groupes abéliens et des groupes hyperboliques, il n'existe pas d'ensembles générateurs de profondeur non-bornée [Bogopolski, 1997; Lehnert, 2009]. Plus généralement, c'est le cas lorsqu'il existe un langage régulier composé de mots géodésiques qui évaluent chaque élément du groupe [Warshall, 2010]. En ce sens, le fait d'avoir une profondeur non-bornée est un obstacle à certaines des propriétés de calcul du groupe. Ce point est approfondi dans [BagnoudBodart, 2022], où ils montrent que le fait d'avoir une profondeur non-bornée implique l'irrationalité de la série de croissance $\mathbb{N}G$ du groupe.

Le premier exemple d'un groupe de type fini avec une profondeur non-bornée était $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ [ClearyTaback, 2005a]. Leur preuve repose sur le fait que dans ce groupe de base unidimensionnel il existent des solutions explicites au TSP, et donc que la métrique des mots sur le produit en couronne admet une expression explicite qui permet des calculs directs. Il n'est pas certain que la propriété d'avoir une profondeur non-bornée soit préservée lorsque l'on considère des produits en couronne plus généraux, puisque l'on ne peut plus s'attendre à obtenir des solutions explicites au TSP. En effet, ce problème est connu pour être NP-complet si $B = \mathbb{Z}^2$.

Nous avons étudié ce problème plus général dans le cas de groupes de lampes avec une profondeur non-bornée, et nous avons prouvé qu'il est toujours possible de trouver des générateurs standards du produit en couronne correspondant qui ont une profondeur non-bornée.

Théorème (= Theorem 2.4.8)**.** *Soit (A, S_A) de profondeur non-bornée et B un groupe de type fini. Alors il existe un ensemble de générateurs fini S_B de B pour lequel $(A \wr B, S_A \cup S_B)$ a une profondeur non-bornée.*

Si nous supposons en plus que le groupe de base B est abélien, ce qui précède est valable pour tout choix de S_B (Proposition 2.4.15). Notons que l'hypothèse que (A, S_A) a une profondeur non-bornée est nécessaire, parce que sinon, dans tout élément du groupe, on pourrait multiplier par des éléments de A pour faire croître la longueur de mots en ignorant la structure de produit en couronne.

La preuve de ces résultats repose sur le fait que nous n'avons pas besoin de connaître les solutions exactes du TSP sur le graphe de Cayley de B, mais seulement leur longueur. Dans le cas des groupes abéliens, nous utilisons le fait qu'il existe des sous-graphes finis de leurs graphes de Cayley qui contiennent des boules arbitrairement grandes, dans lesquels la solution du TSP correspond à un chemin qui est proche d'être hamiltonien (c'est-à-dire qui ne visite qu'un nombre borné de sommets plus d'une fois).

En outre, nous prouvons également que l'on ne peut pas s'attendre à obtenir une profondeur non-bornée pour tout choix d'ensemble générateur standard dans tout produit en couronne $A \wr B$ avec B non abélien.

Théorème (= Theorem 2.5.3)**.** *Soit* (A, S_A) *de profondeur non-bornée, et considérons deux groupes finis* H *et* K *avec des ensembles générateurs* S_H *et* S_K*, respectivement. Alors* $(A \wr (H * K), S_A \cup S_H \cup S_K)$ *n'a pas de profondeur bornée si et seulement si* $\mathscr{H}(H, S_H) + \mathscr{H}(K, S_K) \geq 1$.

Ici, la valeur de $\mathscr{H}(G, S_G)$, pour G un groupe fini, quantifie la longueur d'un cycle minimal de $\mathrm{Cay}(G, S_G)$ par rapport aux chemins minimaux de e_G vers un élément distinct à l'identité (Définition 2.5.2). La condition $\mathscr{H}(H, S_H) + \mathscr{H}(K, S_K) \geq 1$ ci-dessus s'applique en particulier lorsque $H = \mathbb{Z}/n\mathbb{Z}$ et $K = \mathbb{Z}/m\mathbb{Z}$ avec $m, n \geq 2$ et $m + n \geq 10$.

Partie 2 : Marches aléatoires et l'identification du bord de Poisson

Dans la deuxième partie de ce travail, nous étudions le comportement asymptotique des marches aléatoires sur les groupes. Étant donné un groupe dénombrable G et une mesure de probabilité μ sur G, la *μ-marche aléatoire sur G* est la chaîne de Markov $\{w_n\}_{n \geq 0}$ avec un espace d'états G et des probabilités de transition $p(x, y) = \mu(y^{-1}x)$, pour $x, y \in G$. Nous supposons que la distribution initiale est concentrée dans l'identité de G, de sorte que $w_0 = e_G$. Dans le cas des groupes de type fini, une famille naturelle de mesures de probabilité est celle des mesures distribuées uniformément sur un ensemble fini de générateurs symétrique. Les marches aléatoires correspondantes sont appelées *marches aléatoires simples* sur G.

Dans cette thèse, nous étudions le problème d'identification du bord de Poisson des marches aléatoires sur les groupes, que nous discutons ci-dessous.

Le *bord de Poisson* de la marche aléatoire μ sur G est un espace de probabilité qui encode le comportement asymptotique des trajectoires infinies, et elle admet plusieurs définitions équivalentes (voir la Section 3.2). Le bord de Poisson a été utilisée à plusieurs reprises pour prouver des résultats sur des groupes qui ne sont pas liés aux marches aléatoires, comme dans l'approche de Furstenberg des théorèmes de superrigidité [Furstenberg, 1963; Furstenberg, 1971], dans le théorème du sous-groupe normal de Bader-Shalom [BaderShalom, 2006], et pour prouver des estimations de croissance de volume quasi-optimales pour le groupe de croissance intermédiaire de Grigorchuk [ErschlerZheng, 2020].

La moyennabilité d'un groupe est entièrement déterminée par le bord de Poisson des marches aléatoires définies par des mesures non dégénérées (c'est-à-dire telles que leur support génère le groupe en tant que semigroupe). Plus précisément, dans tout groupe non-moyennable, toute marche aléatoire avec des incréments donnés par une mesure non-dégénérée a un bord de Poisson non-trivial [Azencott, 1970, Proposition II.1] (voir aussi la fin de la Section 9 dans [Furstenberg, 1973]). Réciproquement, tout groupe moyennable admet une marche aléatoire définie par une mesure de probabilité supportée sur le groupe entier qui a un bord de Poisson trivial [KaimanovichVershik, 1983; Rosenblatt, 1981]. En raison de ce résultat, les groupes moyennables peuvent être divisés en deux classes. La première est composée de groupes dans lesquels *chaque* marche aléatoire non dégénérée a un bord de Poisson trivial. Ces groupes sont généralement appelés “groupes de Choquet-Deny”, et dans le cas dénombrable, ils constituent précisément la famille des groupes qui n'admettent pas de quotient non trivial où chaque élément non trivial a une classe de conjugaison infinie [FrischHartmanTamuzVahidi Ferdowsi, 2019] (voir Subsection 3.3.1 pour les références aux travaux antérieurs sur ce problème). En particulier, cette équivalence dit qu'un groupe de type fini est Choquet-Deny si et seulement si il est virtuellement nilpotent. La deuxième classe de groupes moyennable est celle formée par les groupes moyennables qui admettent des mesures non dégénérées avec un bord de Poisson non-trivial. D'après Kaimanovich-Vershik, les produits en couronne de la forme $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$, $d \geq 1$, appartiennent à cette famille [KaimanovichVershik, 1983]. De plus, le cas $d \geq 3$ a donné les premiers exemples de groupes moyennables qui admettent des marches aléatoires simples avec un bord de Poisson non-trivial. Des exemples de marches aléatoires non symétriques sur des groupes moyennables avec un bord de Poisson non-trivial avaient été décrits dans [Furstenberg, 1973, dernier paragraphe de la section 9] pour le groupe de Baumslag-Solitar $\mathrm{BS}(1,2)$.

Une quantité clé dans l'étude des bords de Poisson est l'entropie. L'entropie d'une mesure μ sur G est définie par $H(\mu) \coloneqq -\sum_{g \in G} \mu(g) \log(\mu(g))$. Pour les mesures de probabilité avec $H(\mu) < \infty$, le *critère d'entropie* de Kaimanovich-Vershik [KaimanovichVershik, 1983] et Derrienic [Derriennic, 1980] affirme que la μ-marche aléatoire a un bord de Poisson trivial si et seulement si l'*entropie asymptotique* $h(\mu) \coloneqq \lim_{n\to\infty} H(\mu^{*n})/n$ est égale à 0. Ce critère a été étendu à un *critère d'entropie conditionnelle* par Kaimanovich [Kaimanovich, 2000, Theorem 4.6], qui établit si un candidat au bord de Poisson est égal à celui-ci.

Rappelons qu'un groupe est dit localement fini si tout sous-groupe de type fini est fini, et qu'une extension d'un groupe localement fini par un groupe cyclique $\mathbb{Z}$ est appelée un groupe *localement fini par* $\mathbb{Z}$. Une famille particulière d'exemples de ce dernier est celle des produits en couronne $F \wr \mathbb{Z}$, pour F un groupe fini non trivial. Les groupes localement finis par $\mathbb{Z}$ peuvent avoir des diverses propriétés géométriques différentes. Par exemple, ils peuvent avoir des fonctions de Følner à croissance arbitrairement rapide (voir [Erschler, 2003], [Gromov, 2008, Section 8.2] et [OlshanskiiOsin, 2013, Corollaire 1.5]). Dans [BrieusselZheng, 2021, Theorem 1.1], il est prouvé qu'il est possible de trouver des extensions cycliques de groupes localement finis avec une fonction de vitesse, probabilité de retour, une fonction d'entropie, un profil ℓ^p-isopérimétrique et une fonction de compression L_p prescrits. En particulier, les auteurs prouvent que toute fonction suffisamment régulière qui croît au moins exponentiellement peut être réalisée comme la fonction de Følner d'un groupe localement fini par $\mathbb{Z}$, qui admet une marche aléatoire simple avec un bord de Poisson trivial [BrieusselZheng, 2021, Corollary 4.7].

J'ai étudié des marches aléatoires sur le groupe $\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}$, qui est localement fini par $\mathbb{Z}$. Ici, $\mathrm{FSym}(\mathbb{Z})$ est le groupe des permutations de $\mathbb{Z}$ qui sont à support fini. Nous appelons ce groupe un le “mélangeur de réverbères” en faisant une analogie avec les groupes d'allumeur de réverbères, où au lieu d'allumer et d'éteindre les lampes, elles sont permutées. Des marches aléatoires sur $\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}$ ont été étudiées dans [Yadin, 2009], où il est prouvé que la fonction de vitesse marche aléatoire simple pour son ensemble de générateurs standard est asymptotiquement équivalente à $n^{3/4}$. Cela implique que la marche aléatoire simple sur le groupe du mélangeur de réverbères a un bord de Poisson trivial. Dans [ErschlerZheng, 2020, Corollaire 1. 4], il est démontré que la fonction de Følner de $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$, ≥ 1, est asymptotiquement équivalente à n^{n^d}, et la probabilité de retour $\mu^{2n}(e)$ de la marche aléatoire simple est asymptotiquement $\exp\left(-n^{\frac{d}{d+2}}\log^{\frac{2}{d+2}} n\right)$.

Toute marche aléatoire sur $\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}$ induit naturellement une marche aléatoire sur $\mathbb{Z}$ via la projection canonique. Si cette marche aléatoire est transiente et la mesure originale a un premier moment fini, on peut montrer que la première coordonnée de la marche aléatoire associée au groupe de permutation $\mathrm{FSym}(\mathbb{Z})$ se stabilise asymptotiquement à une fonction limite. J'ai prouvé que dans cette situation, cette convergence décrit complètement le bord de Poisson associé.

Théorème (= Theorem 4.1.2)**.** *Considérons une mesure de probabilité μ sur* $\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}$*. Supposons que μ a un premier moment fini et que sa projection sur $\mathbb{Z}$ induit une marche aléatoire transiente. Alors le bord de Poisson de μ est complètement décrit par l'espace des fonctions limites, doté de la mesure harmonique correspondante.*

Le premier exemple d'une marche aléatoire symétrique sur un groupe moyennable avec un bord de Poisson non-trivial a été donnée par Kaimanovich-Vershik [KaimanovichVershik, 1983], qui ont montré que les marches aléatoires simples sur $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$, $d \geq 3$, ont un bord de Poisson non-trivial. Depuis, le problème d'identification du bord de Poisson des produits en couronne $A \wr B$ a été étudié sous diverses conditions de moment sur la mesure de probabilité μ, dans les cas où A est fini ou de type fini, et où $B = \mathbb{Z}^d$ [Erschler, 2011; JamesPeres, 1996; Kaimanovich, 2001; LyonsPeres, 2021a], B un groupe libre non-abélien [KarlssonWoess, 2007], et B un groupe hyperbolique non-élémentaire [Sava, 2010].

Nous pouvons associer à chaque élément du produit en couronne $A \wr B$ une *configuration de lampe* dans $\bigoplus_B A$. Étant donné une mesure de probabilité μ sur $A \wr B$ avec un support fini et qui induit une marche aléatoire transiente sur B, la configuration de lampe se stabilisera le long des trajectoires vers une configuration de lampe limite. Kaimanovich et Vershik ont demandé si, sous les hypothèses précédentes sur μ, l'espace des configurations limites doté de la mesure harmonique correspondante décrit complètement le bord de Poisson de la marche aléatoire de μ sur $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$ [KaimanovichVershik, 1983]. Erschler a répondu positivement à cette question pour $d \geq 5$ et des mesures de probabilité μ avec un troisième moment fini [Erschler, 2011], et Lyons-Peres pour $d \geq 3$ et μ avec un deuxième moment fini [LyonsPeres, 2021a].

Plus généralement, la stabilisation des configurations de lampes se produit également pour des mesures de probabilité avec un support infini qui ont un premier moment fini et qui induisent une marche aléatoire transiente sur $\mathbb{Z}^d$. Kaimanovich [Kaimanovich, 2001, Exemple 3.6.7] et Lyons-Peres [LyonsPeres, 2021a, Section 5] se sont demandé si, pour cette classe plus large de mesures de probabilité, l'espace des configurations limites décrit complètement le bord de Poisson de $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$. En collaboration avec Joshua Frisch, nous donnons une réponse positive à cette question.

Théorème (= Theorem 5.1.6)**.** *Soient A, B des groupes de type fini et μ une mesure de probabilité non-dégénérée sur $A \wr B$. Supposons que μ a un premier moment fini et qu'il induit une* *marche aléatoire transiente sur B. Alors l'espace des configurations de lampes limites A^B dotées de la mesure harmonique correspondante est égal au bord de Poisson de $(A \wr B, \mu)$.*

De plus, nous prouvons le résultat suivant, qui ne pose aucune hypothèse sur les moments de la mesure μ.

Théorème (= Theorem 5.1.3). *Considérons un groupe dénombrable non-trivial A. Soit μ une mesure de probabilité sur $A \wr \mathbb{Z}^d$, $d \geq 1$, avec une entropie finie et telle que les configurations de lampes se stabilisent presque sûrement. Alors le bord de Poisson de $(A \wr \mathbb{Z}^d, \mu)$ est complètement décrit par l'espace $A^{\mathbb{Z}^d}$, doté de la mesure harmonique correspondante.*

Pour que ce résultat tienne, il est nécessaire que nous ayons l'hypothèse de stabilisation des configurations de lampes le long des trajectoires. Ce n'est pas toujours le cas, et il y a des exemples où cela ne se produit pas pour des mesures μ avec un $(1-\varepsilon)$-moment fini, pour tout $\varepsilon > 0$ [Kaimanovich, 1983, Proposition 1.1] (voir aussi [Erschler, 2011, Section 6] et le dernier paragraphe de la Section 5 dans [LyonsPeres, 2021a]). Néanmoins, il est vrai que toute marche aléatoire non-dégénérée sur $A \wr B$ d'entropie finie et une projection transiente sur B a un bord de Poisson non-trivial [Erschler, 2004b, Theorem 3.1]. Dans ce cas, la description du bord de Poisson semble plus difficile, car il n'y a même pas de candidats connus.

L'intérêt de considérer des mesures avec un premier moment possiblement infini vient du fait que de telles mesures sont une partie importante de la théorie des limites de Poisson. La correspondance entre la moyennabilité et l'existence de mesures non-dégénérées avec un bord de Poisson trivial n'est plus vraie si l'on se restreint aux mesures avec un premier moment fini. En effet, le groupe $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^3$ est moyennable, mais toute mesure non-dégénérée avec un premier moment fini (et plus généralement, d'entropie finie) a un bord de Poisson non-trivial. De plus, c'est une conséquence du critère d'entropie que toute mesure avec un premier moment fini sur un groupe de croissance intermédiaire a un bord de Poisson trivial. Néanmoins, la classification des groupes de Choquet-Deny de [FrischHartmanTamuzVahidi Ferdowsi, 2019] implique que chaque groupe de type fini de croissance intermédiaire admet une mesure de probabilité avec un bord de Poisson non-trivial. Ce résultat ne garantit aucun contrôle sur la décroissance de la queue de la mesure. Des mesures de probabilité sur les groupes de croissance intermédiaire avec un α-moment fini, pour $0 < \alpha < 1$, sont construites dans [Erschler, 2004a; ErschlerZheng, 2020] et elles fournissent des estimations de croissance de volume pour les groupes. En particulier, [ErschlerZheng, 2020] l'a utilisé pour donner des bornes inférieures presque optimales pour la croissance du volume du groupe de Grigorchuk.

Les résultats décrivant un bord de Poisson non trivial supposent généralement la finitude d'un moment de μ. Les seuls résultats connus qui décrivent un bord de Poisson non-trivial pour toutes les mesures d'entropie finie sont [ForghaniTiozzo, 2019, Theorem 1.2] pour le semigroupe libre et [ChawlaForghaniFrischTiozzo, 2022] pour les groupes hyperboliques et plus généralement les groupes acylindriquement hyperboliques. En revanche, il y a très peu de résultats qui identifient un bord de Poisson non trivial pour des mesures d'entropie infinie sur les groupes. Une exception que l'on connaît est [ErschlerKaimanovich, 2023, Theorem A], qui décrit le bord de Poisson des groupes ICC pour une classe de mesures issues de la construction de [FrischHartmanTamuzVahidi Ferdowsi, 2019]. En outre, [ForghaniKaimanovich, 2015] décrit le bord de Poisson du semigroupe libre pour des mesures avec un premier moment logarithmique fini sans supposer entropie finie (voir également [ForghaniTiozzo, 2019, Theorem 1.2] et [Forghani, 2015, Theorem 3.6.3]). Il est important de remarquer que même dans le cas d'un semigroupe libre non abélien, c'est un problème ouvert de savoir si l'espace des mots infinis doté de la mesure harmonique correspondante décrit complètement le bord de Poisson pour toutes les mesures d'entropie finie.

Organisation

Le chapitre 1 fournit le contexte nécessaire sur la théorie géométrique des groupes pour cette thèse. Nous discutons de diverses caractérisations de la moyennabilité et rassemblons des résultats sur les propriétés géométriques des produits en couronne. Le reste du manuscrit est divisé en deux parties. La première partie concerne la géométrie des graphes de Cayley des produits en couronne. Le chapitre 2 correspond à une reproduction de l'article [Silva, 2023a]. La deuxième partie de ce manuscrit concerne les marches aléatoires sur les groupes. Dans le chapitre 3, nous rappelons des propriétés basiques des marches aléatoires sur les groupes, nous donnons plusieurs définitions équivalentes du bord de Poisson, et nous rassemblons les résultats connus sur l'identification des bords de Poisson des groupes. Les deux chapitres suivants sont des reproductions d'articles : Le chapitre 4 correspond au prépublication [Silva, 2023b], et le chapitre 5 correspond au prépublication [FrischSilva, 2023] écrit en collaboration avec Joshua Frisch.

Chapter 1

The geometry of groups

This chapter gives an introduction to geometric group theory, with a focus on the most relevant topics and families of groups for this thesis.

In Section 1.1 we recall basic concepts on group theory and the definition of the word metric on a finitely generated group. We also define Cayley graphs and recall basic properties of the growth of groups. Our main references for this section are [Harpe, 2000], [DruţuKapovich, 2018, Chapters 7 and 8], [Löh, 2017, Chapters 2, 3 and 6] and [Mann, 2012]. In Section 1.2 we define wreath products and explain the connection between the word length with respect to standard generating sets and the Traveling Salesperson Problem in the Cayley graph of the base group. Then, in Section 1.3 we give various definitions of amenability, and mention some of its basic properties and connections with other concepts in group theory. Some references for amenability of groups we used are [Ceccherini-SilbersteinCoornaert, 2010, Chapter 4], [DruţuKapovich, 2018, Chapter 18], [Greenleaf, 1969], [Juschenko, 2022, Chapters 1-3; Appendix A], [KerrLi, 2016, Chapter 4], [Löh, 2017, Chapter 9], [Paterson, 1988; Pier, 1984] and [Zimmer, 1978, Chapter 4]. Next, in Section 1.4 we discuss geometric properties of wreath products and lampshufflers that have been studied in the last decades. We make an emphasis in results related to the asymptotic geometry of Cayley graphs and in results regarding random walks on wreath products.

1.1 Geometric group theory

Throughout this thesis we will always work with countable groups and, in most occasions, finitely generated. That is, groups G for which there exists a finite subset $S \subseteq G$ that generates G as a group.

1.1.1 Word metrics and Cayley graphs

A finitely generated group G can be endowed with a metric structure associated with each finite generating set S of G. Let us define the *word length* $|\cdot|_S : G \to G$ by

$$|g|_S \coloneqq \min\{n \geq 0 \mid g = s_1 \cdots s_n \text{ with } s_i \in S \cup S^{-1} \text{ for } i = 1, \dots, n\}, \text{ for } g \in G.$$

The *word metric* $d_S : G \times G \to G$ is defined by $d_S(g,h) := |h^{-1}g|_S$, for $g, h \in G$. As its name suggests, the word metric is indeed a metric on G, and it is invariant under the action of G on itself by left multiplication. The word metric on G can also be defined as the path metric induced by a graph structure on G.

Definition 1.1.1. Let G be a group and let S be a finite generating set of G. The (right) *Cayley graph* $\mathrm{Cay}(G,S)$ is the unlabeled, undirected graph with vertex set G and where edges connect elements g and gs, for $g \in G$ and $s \in S \cup S^{-1}$.

We remark that some definitions of Cayley graphs have oriented edges that are labeled by the generators $s \in S$. In this thesis we will always work with the undirected and unlabeled version above.

The word metric d_S coincides with the path metric of the corresponding Cayley graph $\mathrm{Cay}(G,S)$. That is, the distance $d_S(g,h)$ between elements $g, h \in G$ corresponds to the length of a shortest path connecting g and h in $\mathrm{Cay}(G,S)$, where each edge has unit length.

1.1.2 Growth of groups

One can obtain geometric information of a group by looking at the volume growth of its Cayley graph. The *growth function* $\gamma_{(G,S)} : \mathbb{N} \to \mathbb{R}$ of a group G with respect to a finite generating set S is defined by

$$\gamma_{(G,S)}(n) := |\{g \in G \mid |g|_S \leq n\}|.$$

The *spherical growth function* $\sigma_{(G,S)} : \mathbb{N} \to \mathbb{R}$ is defined by $\sigma_{(G,S)}(0) := 0$ and $\sigma_{(G,S)}(n) := \gamma_{(G,S)}(n) - \gamma_{(G,S)}(n-1)$.

We will omit the subindex "(G,S)" from the growth functions when there is no risk of confusion.

Let $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions. We say that $f \preccurlyeq g$ if there exist $C_1, C_2 > 0$ such that $f(x) \leq C_1 g(C_1 x + C_2) + C_2$ for every $x \in \mathbb{R}_+$. We say that $f \sim g$ if $f \preccurlyeq g$ and $g \preccurlyeq f$.

Note that for every $d \geq 0$ we have $n^d \preccurlyeq n^{d+1}$ and $n^{d+1} \not\preccurlyeq n^d$. In contrast, for any values $\lambda_1, \lambda_2 > 1$, we have $\lambda_1^n \sim \lambda_2^n$.

If we have two generating sets S_1 and S_2 of the same group G, the associated growth functions $\gamma_{(G,S_1)}$ and $\gamma_{(G,S_2)}$ will not be equal. However, their large-scale behavior is the same in the sense that $\gamma_{(G,S_1)}(n) \sim \gamma_{(G,S_2)}(n)$.

For any finitely generated group G it holds that $\gamma(n) \preccurlyeq \exp(n)$. Indeed, the growth function of G with respect to S is bounded above by the growth function of the free group $F(S)$ with respect to S, which is equivalent to $\exp(n)$. However, it is not always the case that $\gamma(n) \sim \exp(n)$. We distinguish the three following situations: A finitely generated group G is said to have

- *exponential growth* if $\gamma(n) \sim \exp(n)$,
- *polynomial growth* if there exists $d \geq 0$ such that $\gamma(n) \preccurlyeq n^d$, and
- *intermediate growth* if it is not of exponential growth nor of polynomial growth.

Historically, the study of growth on finitely generated groups can be traced back to Schwarz, who showed that the type of volume growth of the universal cover $\widetilde{M}$ of a compact Riemannian manifold M coincides with the type of growth of the fundamental group $\pi_1(M)$ [Švarc, 1955]. Later, Milnor proved that if M is a compact Riemannian manifold with negative sectional curvature, then the fundamental group $\pi_1(M)$ has exponential growth [Milnor, 1968a]. In the same paper, Milnor also proved that if M is a complete d-dimensional Riemannian manifold whose mean curvature tensor is everywhere positive semi-definite, then the growth function γ of $\pi_1(M)$ satisfies $\gamma(n) \preccurlyeq n^d$.

Every finitely generated nilpotent group has polynomial growth [Wolf, 1968]. Furthermore, [Bass, 1972] and [Guivarch, 1973] proved that every finitely generated nilpotent group has a growth function $\gamma(n) \sim n^d$, where $d \in \mathbb{N}$ is an integer determined by the lower central series of G. Gromov's Theorem of polynomial growth states that every finitely generated group of polynomial growth has a nilpotent subgroup of finite index [Gromov, 1981b]. In particular, these two results imply that it is not possible for a finitely generated group to have a growth function equivalent to n^α for $\alpha \in \mathbb{R}\backslash\mathbb{Z}$.

There are many families of groups where there is a dichotomy between polynomial and exponential growth. This is the case of the Milnor-Wolf theorem, which states that finitely generated solvable groups are either virtually nilpotent or have exponential growth [Milnor, 1968a; Wolf, 1968], and of the Tits alternative, which says that if $\mathbb{K}$ is a field and $n \geq 1$, then every finitely generated subgroup of $\mathrm{GL}(n, \mathbb{K})$ is either virtually solvable or it contains a free subgroup of rank 2 [Tits, 1972]. The existence of finitely generated groups of intermediate growth was asked by Milnor in [Milnor, 1968b], and the first constructions of such groups were obtained by Grigorchuk in [Grigorchuk, 1984; Grigorchuk, 1985].

The growth function of a group can also be studied via its associated power series. Let G be a group and S a finite generating set. The *growth series* of G with respect to S is the power series $\Gamma_{(G,S)} : \mathbb{C} \to \mathbb{C}$ defined by

$$\Gamma_{(G,S)}(z) \coloneqq \sum_{n\geq 0} \gamma_{(G,S)}(n) z^n, \text{ for } z \in \mathbb{C}.$$

We say that G has:

- *rational growth* with respect to S if $\Gamma_{(G,S)}$ is a rational function (i.e. the quotient of two polynomials),
- *algebraic growth* with respect to S if $\Gamma_{(G,S)}$ satisfies a polynomial equation with coefficients in the field of polynomials $\mathbb{C}[z]$, and
- *transcendental growth* with respect to S if it does not have algebraic growth.

It is known that any virtually abelian group has rational growth with respect to any generating set [Benson, 1983]. It is also known that hyperbolic groups have rational growth for every generating set [Gromov, 1987, Theorem 8.5.D] (see also [Cannon, 1984, Theorem 7], [GhysHarpe, 1990, Théorème 1.39] and [EpsteinCannonHoltLevyPatersonThurston, 1992, Theorem 3.4.5]).

There are some combinatorial restrictions that prevent a group from having rational growth. Namely, If a power series with integer coefficients is rational, then its coefficients must grow either polynomially or exponentially. This implies that groups of intermediate growth have irrational growth with respect to every generating set [Mann, 2012, Theorem 15.1]. Another consequence of having a rational growth series is that, if the group is recursively presented, the word problem on the group must be decidable [Mann, 2012, Proposition 1.6]. In particular, any finitely presented group with undecidable word problem must have irrational growth with respect to every generating set.

Aside from what is mentioned above, it is in general an open problem to understand under which conditions a group has rational growth with respect to a given generating set. This is the case even for 2-step nilpotent groups. It is known that in every finitely generated 2-step nilpotent group with an infinite cyclic derived subgroup there is some generating set with rational growth [Stoll, 1996, Theorem A]. In particular for the discrete Heisenberg group $H_3(\mathbb{Z})$, the rationality of its growth series was first established for its standard generators [Benson, 1987; Shapiro, 1989] and recently to every generating set [DuchinShapiro, 2019, Theorem 1]. The latter is not always the case: it is shown in [Stoll, 1996, Theorems A, B and C] that higher-rank Heisenberg groups $H_n(\mathbb{Z})$, $n \geq 5$, have some generating sets with rational growth series and other generating sets with transcendental growth series.

1.2 Wreath products

The most important family of groups for the results of this thesis are wreath products of groups. We now give their definition and explain basic aspects of their geometry.

For two groups A and B, the direct sum $\bigoplus_B A$ is the group of finitely supported functions $f : B \to A$ endowed with the operation $\oplus$ of componentwise multiplication. We denote by $\operatorname{supp}(f)$ the finite subset of B to which f assigns values distinct from e_A.

Definition 1.2.1. The *wreath product* $A \wr B$ of the groups A and B is defined as the semidirect product $\bigoplus_B A \rtimes B$.

Here, the group B acts on the direct sum $\bigoplus_B A$ from the left by translations. That is, for $f : B \to A$, and any $b \in B$ we have

$$(b \cdot f)(x) = f(b^{-1}x),\ x \in B.$$

Remark 1.2.2. The group $A \wr B$ is also referred to in the literature as the *restricted* wreath product of A and B. This is done to distinguish $A \wr B$ from the *unrestricted wreath product* $A \wr\wr B := \Pi_B A \rtimes B$, where $\Pi_B A$ is the group of functions $f : B \to A$ with a possibly infinite support, endowed with the operation of componentwise multiplication. If the group B is infinite and A is non-trivial, then $A \wr\wr B$ is uncountable and we will not consider it in this thesis.

If the group B acts on the left on some set X, then it is possible to define the *permutational wreath product* $A \wr_X B$ is defined as $\bigoplus_X A \rtimes B$, where the action of B on X induces an action by translations on $\bigoplus_X A$. If we consider $X = B$ endowed with the left action by multiplication, we recover the definition of $A \wr B$ above. However, we remark that the geometric and algebraic properties of permutational wreath products can be quite different from those of wreath products. For example, whereas the wreath product of a non-trivial group A with an infinite group B always has exponential growth, it is possible for permutational wreath products to have intermediate growth [BartholdiErschler, 2012; BartholdiErschler, 2014a].

Wreath products $A \wr B$ are in general infinitely presented, even if the groups A and B are finitely presented. Indeed, $A \wr B$ will be finitely presented only if A is trivial and B is finitely presented, or if A is finitely presented and B is finite [Baumslag, 1961]. Additionally, it is proved in [Baumslag, 2005] that the wreath product $A \wr B$ of finitely generated groups A and $B$is recursively presentable if and only if both A and B are recursively presentable, and, if G is non-trivial, then either A has a solvable word problem or B is abelian.

1.2.1 The word length and the TSP

We now describe in more detail the metric structure of Cayley graphs of $A \wr B$ with respect to standard generators.

Elements of $A \wr B$ can be expressed as a tuple (f, b), where $f : B \to A$ is a finitely supported function and $b \in B$. The product between two elements elements (f, b), $(f', b') \in A \wr B$ is given by

$$(f, b) \cdot (f', b') = (f \oplus (b \cdot f'), bb').$$

There is a natural embedding of B into $A \wr B$ via the mapping

$$\begin{aligned} B &\to A \wr B \\ b &\mapsto (\mathbb{1}, b), \end{aligned}$$

where $\mathbb{1}(x) = e_A$ for any $x \in B$. Similarly, we can embed A into $A \wr B$ via the mapping

$$\begin{aligned} A &\to A \wr B \\ a &\mapsto (\delta^a_{e_B}, e_B), \end{aligned}$$

where $\delta^a_{e_B}(e_B) = a$ and $\delta^a_{e_B}(x) = e_A$ for any $x \neq e_B$.

In particular, if we consider finite symmetric generating sets S_A and S_B of A and B, respectively, their copies inside $A \wr B$ through the above embeddings generate the entire group $A \wr B$. We call $S_{\text{std}} \coloneqq S_A \cup S_B$ the *standard generating set* for $A \wr B$ associated with the generators S_A and S_B.

The word length in $A \wr B$ with respect to a standard generating set can be expressed in terms of the word length in A and a combinatorial problem in the Cayley graph of B, namely the *Traveling Salesperson Problem*. More precisely, let us denote by $\mathrm{TS}(b, b', f)$ the length of a path of minimal length in $\mathrm{Cay}(B, S_B)$ that starts at b, finishes at b' and visits all vertices of $\mathrm{supp}(f)$. The notation TS stands for the interpretation of said path as a solution to the Traveling Salesperson Problem (TSP). We also write $\mathrm{TS}(b, b', F)$ to denote a path of minimal length in $\mathrm{Cay}(B, S_B)$ starting at b, finishing at b' and visiting all vertices from a finite subset $F \subseteq B$.

Theorem 1.2.3 ([Parry, 1992, Theorem 1.2]). *The word length of an element $g = (f, x) \in A \wr B$ with respect to S_{std} is given by*

$$|g|_{S_{\text{std}}} = \sum_{y \in \operatorname{supp}(f)} |f(y)|_{S_A} + \operatorname{TS}(e_B, x, f).$$

The geometric interpretation of this description of the word length is the following. Consider $g = (f, x) \in A \wr B$ and write it as a product of generators of S_{std},

$$g = a_0 b_1 a_1 b_2 a_2 \cdots b_m a_m,$$

with $m \geq 0$, $a_0, \ldots, a_m \in A$, and $b_1, \ldots, b_m \in B$. In particular, it holds that

$$f = a_0 (b_1 \cdot a_1)(b_1 b_2 \cdot a_2) \cdots (b_1 b_2 \cdots b_m \cdot a_m),$$

so that $\operatorname{supp}(f) \subseteq \{e_B, b_1, b_1 b_2, \ldots, b_1 b_2 \cdots b_m\}$, and $x = b_1 b_2 \cdots b_m$. This factorization of g into generators of S_{std} can be interpreted as a path in the Cayley graph $\operatorname{Cay}(B, S_B)$ which begins at e_B, visits all vertices in $\operatorname{supp}(f)$ while generating the appropriate group element for f in each one, and ends at $b_1 b_2 \cdots b_m$. This is the reason behind the name "lamplighter group": one can think of the Cayley graph $\operatorname{Cay}(B, S_B)$ as a street with lamps at every vertex, each of which can be in a different state given by an element of A. Then a word in S_{std} evaluating to an element $g = (f, x) \in A \wr B$ corresponds to a path from the origin e_B to x, which passes through all vertices of $\operatorname{supp}(f)$ and at each one of them uses the generators of S_A in order to reach the value of f there. We refer to x as the *position of the lamplighter*, and to f as the *lamps configuration*.

Parry used Theorem 1.2.3 while studying the growth series of wreath products of the form $A \wr F$, where F is a free product of finitely many copies of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$. He proved that if A has rational growth with respect to some generating set and if F is virtually $\mathbb{Z}$ then $A \wr F$ has rational growth [Parry, 1992, Corollary 3.3], and that if F is not virtually $\mathbb{Z}$ then $A \wr F$ has algebraic but not rational growth [Parry, 1992, Theorem 3.7], both for standard generating sets.

1.3 Amenable groups

The study of amenable groups was initiated by von Neumann in his study of the Hausdorff-Banach-Tarski paradox [Neumann, 1929], where he originally used the word "meßbar" (which translates to "measurable" in English) to refer to this family of groups. Below, we define amenable groups via the existence of invariant means and then present and comment on equivalent definitions that are relevant for the topics of this thesis, most notably those related to random walks on groups. There are several other equivalent definitions of amenability that we do not mention here, and they can be found in the references given at the beginning of this chapter.

Let G be a group, and consider the space $\ell^\infty(G, \mathbb{R})$ of bounded functions $G \to \mathbb{R}$. There is a natural left action of G on $\ell^\infty(G,\mathbb{R})$ given by

$$(g \cdot f)(x) = f(g^{-1}x), \text{ for } f \in \ell^\infty(G,\mathbb{R}) \text{ and } g, x \in G.$$

Let us denote by $\mathbb{1} \in \ell^\infty(G,\mathbb{R})$ the constant function that assigns the value 1 to every element of G.

We say that an $\mathbb{R}$-linear map $\mathfrak{m} : \ell^\infty(G,\mathbb{R}) \to \mathbb{R}$ is a *mean* if it satisfies

— $\mathfrak{m}(f) \geq 0$ for every $f \in \ell^\infty(G,\mathbb{R})$ that satisfies $f \geq 0$ pointwise, and

— $\mathfrak{m}(\mathbb{1}) = 1$.

A mean $\mathfrak{m}$ is said to be *left-invariant* if $\mathfrak{m}(g \cdot f) = \mathfrak{m}(f)$ for all $f \in \ell^\infty(G,\mathbb{R})$ and $g \in G$.

Definition 1.3.1. A group G is *amenable* if there exists a left invariant mean on $\ell^\infty(G,\mathbb{R})$.

The family of amenable groups includes finite groups, abelian groups and more generally solvable groups. Furthermore, amenability is preserved under various group operations:

— If G is amenable and $H \leqslant G$ is a subgroup of G, then H is amenable.

— If G is amenable, and $\varphi : G \to K$ is a homomorphism into a group K, then $\varphi(G)$ is amenable. In particular, for every normal subgroup $N \lhd G$, the quotient group G/N is amenable.

— Let $N \lhd G$ be a normal subgroup of G such that N and G/N are amenable. Then G is amenable.

— Let $\{H_i\}_{i\in I}$ be a directed system of amenable subgroups $H_i \leqslant G$ such that $G = \bigcup_{i\in I} H_i$. Then G is amenable.

For the proofs of these facts we refer to [Greenleaf, 1969, Theorems 1.2.4 - 1.2.7].

The last item above shows that amenability is a local property of groups: a countable group is amenable if and only if each of its finitely generated subgroups is amenable. In particular, locally finite groups, that is, groups where every finitely generated subgroup is finite, are amenable. Hence, if A and B are amenable groups, then so is $A \wr B := \bigoplus_B A \rtimes B$. Similarly, if H is an amenable group, then the group $\mathrm{FSym}(H)$ of finitary permutations of H is amenable and so is the extension $\mathrm{FSym}(H) \rtimes H$.

The connection between amenability and random walks on groups goes back to Harry Kesten's PhD thesis [Kesten, 1959b], which is commonly attributed as the origin of the area of random walks on (infinite, non-abelian) groups. By studying the spectral radius of the Markov operator associated with a random walk on a group, he obtained the following criterion for amenability in [Kesten, 1959a].

Theorem 1.3.2 (Kesten's criterion)**.** *Let G be a finitely generated group, and let μ be a finitely supported symmetric probability measure on G. Consider the* spectral radius

$$\rho := \limsup_{n\to\infty} \mu^{*2n}(e_G)^{1/2n}$$

of the μ-random walk on G. Then G is amenable if and only if $\rho = 1$.

Note that $\mu^{*2n}(e_G)$ is the probability of returning to the identity after $2n$ steps of the random walk on G with step distribution μ.

Non-abelian free groups are examples of non-amenable groups, and so is any group that contains a non-abelian free subgroup, since amenability is preserved by passing to a subgroup. The question of whether every non-amenable group contains a non-abelian free subgroup was asked by Day at the end of Section 4 in [Day, 1957]. Kesten mentions in [Kesten, 1959b, Section 5] that it may be possible that torsion groups with a spectral radius $\rho < 1$ exist, which would thus be non-amenable thanks to his criterion. The existence of infinite finitely generated torsion groups was proved by [NovikovAdjan, 1968], who showed that free Burnside groups of sufficiently large exponent are infinite. Developing on their method, [Olshanskii, 1980b] gave the first example of a non-amenable group with no free non-abelian subgroups, and in [Olshanskii, 1980a] constructed for every sufficiently large prime p, non-amenable groups where every non-trivial proper subgroup is cyclic of order p. Infinite groups with the latter property are called "Tarski monsters". [Adyan, 1982] showed that free Burnside groups of sufficiently large odd exponent are non-amenable. Both of these results prove non-amenability by using a *cogrowth criterion*, which was proved independently in [Grigorchuk, 1980] and in [Cohen, 1982]. This criterion states that the amenability of a finitely generated group is characterized by the maximality of the group cogrowth rate with respect to a finite symmetric generating set. More precisely, let G be a finitely generated group and S a finite generating set. Denote by N the kernel of the quotient map $F(S) \to G$, where $F(S)$ is the free group with a free generating set S. Let $\gamma_n := |\{g \in N \mid |g|_S \leq n\}|$ be the number of elements in N with word length at most n.

Theorem 1.3.3 (Cogrowth criterion)**.** *The group G is amenable if and only if $\lim_{n\to\infty} \gamma_n^{1/n} = 2|S| - 1$.*

The original proofs in [Grigorchuk, 1980] and [Cohen, 1982] of this result are based on Kesten's criterion above (Theorem 1.3.2). An alternative combinatorial proof is presented in [Bartholdi, 1999].

The amenability of a group is also characterized by the dynamical properties of the actions of G on compact spaces.

Theorem 1.3.4. *A group G is amenable if and only if every continuous action of G on a compact space X admits an invariant regular Borel probability measure.*

This is the definition for amenability given in [Zimmer, 1978, Definition 4.1.1]. For a proof of the equivalence with the other definitions we have given, we refer to [KerrLi, 2016, Theorem 4.4.iii)]. Using this characterization of amenability, it is possible to prove that non-amenable groups admit non-constant bounded μ-harmonic functions for each probability measure μ on G whose support generates G (see Corollary 3.2.14). This result was first proved in the corollary immediately after [Azencott, 1970, Proposition II.1] and in [Furstenberg, 1973, Section 9]. Furthermore, the existence of non-degenerate probability measures on G that only admit constant bounded μ-harmonic functions implies that G is amenable [Rosenblatt, 1981, Theorem 1.10] and [KaimanovichVershik, 1983, Theorem 4.4] (see Theorem 3.3.1). For this direction of the equivalence, it is used that the amenability of a group is completely determined by the existence of "almost invariant" finite subsets of the group. This is known as Følner's characterization of amenability, which was originally proved in [Følner, 1955, Main Theorem].

Theorem 1.3.5 (Følner's criterion)**.** *A countable group G is amenable if and only if for each $\varepsilon > 0$ and every finite subset $K \subseteq G$, there exists a finite subset $F \subseteq G$ such that $|KF \triangle F| \leq \varepsilon|F|$. We say that the set F is a (K, ε)-*Følner set.

The family of *elementary amenable* groups was introduced in [Day, 1957] and it consists of the smallest class of groups that contains finite groups and abelian groups while also being closed under direct limits, group extensions, subgroups, and quotients. Day asked the question of whether every amenable group is elementary amenable. This question was answered negatively by [Grigorchuk, 1984; Grigorchuk, 1985], who provided the first example of a finitely generated and infinite group with intermediate growth. It can be shown that if a group G has subexponential growth, then balls of sufficiently large radii provide Følner sets, and hence every group of subexponential growth is amenable. In addition, every elementary amenable group is either virtually nilpotent, and hence has polynomial growth, or it contains a non-abelian free subsemigroup, and so it has exponential growth [Chou, 1980, Theorem 3.2]. Consequently, any finitely generated group with intermediate growth is amenable but not elementary amenable, and hence Grigorchuk's group provides a negative answer to Day's question.

One can also consider the class of *subexponentially amenable* groups, which is the smallest class of groups that contains all groups of subexponential growth and that is closed under direct limits, group extensions, subgroups, and quotients. Every subexponentially amenable group is amenable, and it was asked in [Grigorchuk, 1998] whether every amenable group is subexponentially amenable. A possible counterexample, which came to be known as the *Basilica group*, was suggested in [GrigorchukŻuk, 2002]. The Basilica group is two-generated, and it is defined via its action on a rooted binary tree. Grigorchuk and Żuk proved that it is not subexponentially amenable, and its amenability was established in [BartholdiVirág, 2005], thus providing the first example of an amenable non-subexponentially amenable group. Using a notion of a "self-similar random walk", Bartholdi and Virág showed that the Basilica group admits a finitely supported probability measure that induces a random walk with zero speed. This argument was later simplified by [Kaimanovich, 2005], who worked with the asymptotic entropy of the random walk instead of the speed. Proving the vanishing of the asymptotic entropy, or, alternatively, the speed, of a symmetric probability measure is equivalent to showing that the associated Poisson boundary is trivial; see Section 3.2. This method could be applied to prove the amenability of a wider range of groups acting on rooted trees, such as groups generated by bounded automata [BartholdiKaimanovichNekrashevych, 2010] and more generally to groups generated by linear activity automata [AmirAngelVirág, 2013]. It is a result of [Sidki, 2004] that any group generated by a polynomial activity automaton does not contains non-abelian free groups, and it is currently an open question whether all such groups are amenable.

It was proved in [JuschenkoMonod, 2013] that the topological full group of a Cantor minimal dynamical system is amenable. Any such group has a simple and finitely generated derived subgroup [Matui, 2006], and due to the previous result, they provided the first examples of simple finitely generated and infinite amenable groups. An alternative proof of this result is due to [Matte Bon, 2014], who showed the vanishing of the asymptotic entropy of symmetric finitely supported random walks on topological full groups of low-complexity subshifts without isolated periodic points. Thus, these groups are amenable, and by Matui's result their derived subgroup provides a finitely generated simple amenable group. A unified approach to proving amenability that includes all known examples of amenable groups is developed via the notion of "extensive amenability" in [JuschenkoMatte BonMonodSalle, 2018; JuschenkoNekrashevychSalle, 2016]. In [Nekrashevych, 2018] it is shown that there exist finitely generated simple Burnside groups of intermediate growth, with a construction via subgroups of topological full groups of some minimal actions of the infinite dihedral group on the Cantor set.

1.4 Wreath products and lampshufflers in the literature

We first give a brief historical account of the origins of wreath products, based in the exposition of [Kerber, 1971]. The first constructions of wreath products (of finite groups) go back to [Cauchy, 1844; Netto, 1882; Radzig, 1895], who used them in the process of constructing p-Sylow subgroups of symmetric groups. Wreath products later appeared in [Loewy, 1927; Neumann, 1933; Scholz, 1930], and representations of wreath products were studied in [Specht, 1932; Specht, 1933; Young, 1930]. The origin of the name "wreath product" is due to [Pólya, 1937], who called it "Kranzgruppe" in German. One of the first results about general wreath products is the Krasner–Kaloujnine universal embedding theorem, which states that any group extension of a normal subgroup A with quotient B embeds into the *unrestricted* wreath product $A \wr\wr B$ (see Remark 1.2.2 for the distinction between restricted and unrestricted wreath products) [KrasnerKaloujnine, 1950; KrasnerKaloujnine, 1951a; KrasnerKaloujnine, 1951b].

Let us now mention the geometric properties of wreath products that have been studied in the past decades.

A quasi-isometric classification of wreath products $F \wr H$, for F finite and H virtually $\mathbb{Z}$ has been proved in [EskinFisherWhyte, 2007; EskinFisherWhyte, 2012; EskinFisherWhyte, 2013], and for F finite and H one-ended in [GenevoisTessera, 2021]. The quasi-isometry classes of some generalizations of wreath products to so-called "halo products" of groups, that in particular include the lampshuffler groups $\mathrm{FSym}(H) \rtimes H$, are studied in [GenevoisTessera, 2024]. Dehn functions of Baumslag's metabelian groups into which lamplighter groups embed are studied in [CornulierTessera, 2010; KassabovRiley, 2012]. The existence of infinitely many twisted conjugacy classes with respect to an automorphism of the wreath product is studied in [GonçalvesWong, 2006; TabackWong, 2007; TabackWong, 2011]. The distortion of embeddings of wreath products into L^p and Hilbert spaces have been studied in [ArzhantsevaGubaSapir, 2006; BaudierMotakisSchlumprechtZsák, 2021; BaudierMotakisSchlumprechtZsák, 2022; Genevois, 2017; Genevois, 2022; Li, 2010; NaorPeres, 2008; NaorPeres, 2011; StalderValette, 2007; Tessera, 2011; Tessera, 2012]. Distorted functions of subgroups of wreath products have been studied in [DavisOlshanskii, 2011], where it is proved that the distortion function of every finitely generated subgroup of $A \wr \mathbb{Z}$, for A a finitely generated abelian group, is a polynomial, whereas [Riley, 2022] shows that $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ and $\mathbb{Z} \wr F_2$ contain exponentially distorted subgroups. The distortion of subgroups of the form $A' \wr B' \leqslant A \wr B$ for subgroups $A' \leqslant A$ and $B' \leqslant B$ is studied in [BurilloLópez-Platón, 2015]. Properties of words and paths in the Cayley graphs of lamplighter groups $\mathbb{Z}/n\mathbb{Z} \wr \mathbb{Z}$, $n \geq 2$, have been studied, such as the non-existence of regular languages of geodesics and other language-theoretical properties in [ClearyElderTaback, 2006], the existence of dead ends with arbitrarily large depth [ClearyTaback, 2005a; ClearyTaback, 2005b] and the decomposition of group elements into a product of a bounded number of palindromic words in [RileySale, 2014]. Isomorphisms between wreath products as well as other algebraic properties of wreath products are studied in [Neumann, 1964], and morphisms and automorphisms of wreath products are studied in [BradfordFournier-Facio, 2022; GenevoisTessera, 2022; MimuraSako, 2021; Mohammadi Hassanabadi, 1978].

It is shown in [Vassileva, 2012] that the Magnus embedding of free solvable groups into a wreath product is a quasi-isometry. In [Sale, 2015] it is furthermore proved that the Magnus embedding is 2-Lipschitz, and this is applied to obtain lower bounds on the L^p compression exponents of free solvable groups. The distortion of natural embeddings of wreath products into other groups has also been studied: it is proved in [Cleary, 2006] that $\mathbb{Z} \wr \mathbb{Z}$ embeds into Thompson's group F as an undistorted group, whereas the embedding of $\mathbb{Z} \wr \mathbb{Z}$ in Baumslag's metabelian group is at least exponentially distorted.

Woess asked in [Woess, 1991] whether there exist locally-finite vertex-transitive graphs that are not quasi-isometric to a Cayley graph. The family of Diestel-Leader graphs $\mathrm{DL}(p,q)$, $p,q \geq 2$, defined as the horocyclic product of a $(p{+}1)$-regular tree and a $(q{+}1)$-regular tree, was introduced in [DiestelLeader, 2001] as possible examples of such graphs. When $p = q$, the Diestel-Leader graph $\mathrm{DL}(p,p)$ coincides with a Cayley graph of a lamplighter group $F \wr \mathbb{Z}$, where F is a finite group of order p. In particular, the graphs $\mathrm{DL}(p,p)$, $p \geq 2$, are amenable. In contrast, if $p \neq q$ the graph $\mathrm{DL}(p,q)$ is non-amenable and it was proved by [EskinFisherWhyte, 2007; EskinFisherWhyte, 2012; EskinFisherWhyte, 2013] that they are in fact not quasi-isometric to any Cayley graph, thus answering Woess' question. Metric properties of the horocyclic product of three or more regular trees are studied in [AmchislavskaRiley, 2015; BartholdiNeuhauserWoess, 2008; SteinTaback, 2013].

The lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is realized as a group defined by a two-state automaton in [GrigorchukŻuk, 2001], and this structure is used to to compute the spectrum and spectral measures of the simple random walk on it. This is then used in [DicksSchick, 2002; Grigorchuk-LinnellSchickŻuk, 2000] to produce a counterexample to the strong version of the Atiyah conjecture about the range of L^2-Betti numbers. More precisely, they constructed a 7-dimensional closed Riemannian manifold whose fundamental group is an HNN-extension of $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$, whose third L^2-Betti number is $1/3$. The spectrum and spectral measures of simple random walks on $\mathbb{Z}/n\mathbb{Z} \wr \mathbb{Z}$, for $n \geq 2$, and more generally of Diestel-Leader graphs are studied in [BartholdiWoess, 2005], where in addition the asymptotic decay of the return probability to the origin after $2n$ steps is determined. The description of spectral measures on wreath products is generalized for groups $F \wr \mathbb{Z}$, for F a non-trivial finite group, in [KambitesSilvaSteinberg, 2006]. Realizations of wreath products of the form $A \wr \mathbb{Z}$, where A is a non-trivial finite abelian group, by other types of automata have also been studied, e.g. for reversible automata [Francoeur, 2023; SkipperSteinberg, 2020], reset automata [SilvaSteinberg, 2005; Yang, 2021]. In [Woryna, 2013] it is shown that wreath products $A \wr \mathbb{Z}$ for A a non-trivial abelian group can be defined via "self-similar automaton over a changing alphabet", which is a generalization of Mealy automata introduced in the same paper.

Positive harmonic functions and the Martin boundary of lamplighter groups and Diestel-Leader graphs are studied in [BrofferioWoess, 2005; BrofferioWoess, 2006; Woess, 2005]. The return probability on wreath products is studied in [PittetSaloff-Coste, 2002; Revelle, 2003a; Varopoulos, 1983]. The asymptotic behavior of the drift function for simple random walks on (iterated) wreath products of the form $(\cdots(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z})\cdots) \wr \mathbb{Z}$, and other iterated wreath products using $\mathbb{Z}^2$ instead of $\mathbb{Z}$, is studied in [Dyubina, 1999; Erschler, 2003; Erschler Dyubina, 2004]. It is proved in [Gilch, 2008; Gilch, 2009] that for any random walk with a finite first moment on $\mathbb{Z}/2\mathbb{Z} \wr G$ and with transient projection to G, the linear rate of escape is strictly bigger than that of its projection to G, excluding some degenerate cases when $G = \mathbb{Z}$. A law of iterated logarithm on wreath products $F \wr \mathbb{Z}$, for F finite, is established in [Revelle, 2003b]. The minimal possible growth of non-constant harmonic functions [BenjaminiDuminil-CopinKozmaYadin, 2017]. The instability of recurrent subsets in wreath products is studied in [BenjaminiRevelle, 2011]. Biased random walks have been studied on lamplighter groups $F \wr \mathbb{Z}$ for F a non-trivial finite group. These are Markov chains on the Cayley graphs of the group that have a nearest-neighbor transition probabilities that assign a different probability to neighbors closer to the identity than those further away from it. In [LyonsPemantlePeres, 1996; Revelle, 2001] it is proved that inward biased random walks drift faster from the origin than the simple random walk, and outward biased random walks drift slower to infinity than the simple random walk. Properties of random walks on finite wreath products have also been studied, such as the mixing and relaxation time [HäggströmJonasson, 1997; KomjáthyPeres, 2016; PeresRevelle, 2004] and late points [DemboDingMillerPeres, 2019; MillerSousi, 2017].

Permutational wreath products play an important role in the study of self-similar groups; see e.g. [Nekrashevych, 2005, Section 1.4]. We also remark that permutational wreath products may have geometric properties that are different from those of usual wreath products. For example, in [BartholdiErschler, 2012] it is shown that among these groups one can find groups of intermediate growth, which provided the first examples of such groups for which the exact asymptotic growth could be calculated. This is studied in further detail in [BartholdiErschler, 2014a], where permutational wreath products are used to realize groups with a given growth function f, for every function f that grows uniformly faster than $\exp(n^\alpha)$, where $\alpha \approx 0.7674$ is a fixed constant. In [BartholdiErschler, 2014b] it is shown that every countable group which does not contain a finitely generated subgroup of exponential growth embeds into a finitely generated group of subexponential growth, by using (unrestricted, permutational) wreath products. Permutational wreath products are used in [BartholdiErschler, 2017] to produce the first examples of groups of exponential growth for which every finitely supported (possibly non-symmetric and degenerate) probability measure has a trivial Poisson boundary. We also refer to [Bartholdi, 2017] for an exposition of these results together with other applications of wreath products. Wreath products also appear in the theory of dynamical systems, representation theory and operators algebras. It is proved in [MonodOzawa, 2010] that wreath products of abelian groups with non-amenable groups are non-unitarizable, and this same family of groups has been shown to provide examples of non-amenable groups that are not strongly Ulam stable [Alpeev, 2023]. The Haagerup property for wreath products is studied in [CornulierStalderValette, 2008; CornulierStalderValette, 2012; Genevois, 2022]. Fixed point properties for actions of wreath products are studied in [CornulierKar, 2011; Genevois, 2022; LeemannSchneeberger, 2022a; LeemannSchneeberger, 2022b; Osajda, 2018]. In some recent papers [ChifanDavisDrimbe, 2022; ChifanDavisDrimbe, 2023; ChifanIoanaOsinSun, 2023a; ChifanIoanaOsinSun, 2023b; ChifanPopaSizemore, 2012] the concept of "wreath-like products of groups" is used to refer to any group W that is an extension of the form

$$1 \to \bigoplus_X A \to W \to B \to 1,$$

where A and B are groups and X is a set of indices on which B acts on the left. The latter action induces an action of B on $\bigoplus_X A$. This family of groups includes wreath products and permutational wreath products.

1.4.1 Lampshufflers and extensions of the finitary symmetric group

Another family of groups that we study in this thesis are "lampshuffler groups", which are defined as $\mathrm{FSym}_{\mathrm{ext}}(H) \coloneqq \mathrm{FSym}(H) \rtimes H$, for H finitely generated, where $\mathrm{FSym}(H)$ is the group of finitely supported permutations of H (see also Section 4.2 in Chapter 4). We now mention some of the geometric properties of these groups that have been studied in the past decades. The lampshuffler group $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ is considered in [VershikGordon, 1997, Section 2.3] as an example of a finitely generated group that is locally embeddable in the class of finite groups (LEF) that is not residually finite (in the same paper it is mentioned that this example goes back to Vershik's Doctor of Sciences thesis [Vershik, 1973]). Additionally, the group $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ was used in [Stëpin, 1983] as an example of a finitely generated non-residually finite group which admits a freely approximable action. This is in contrast with [Stëpin, 1983, Theorem 1], which states that any finitely presented group that admits such an action must be residually finite. The subgroup $\mathrm{FAlt}(\mathbb{Z}) \rtimes \mathbb{Z}$ of the lampshuffler group $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$, where $\mathrm{FAlt}(\mathbb{Z})$ stands for the group of finitary even permutations of $\mathbb{Z}$, is used in [Chou, 1980, Example 2] to illustrate explicitly an example of the existence of free subsemigroups in elementary amenable groups that are not virtually solvable. In [ElekSzabó, 2006, Theorem 3] it is shown that if H is an infinite, hyperbolic, residually finite group with Kazhdan's property (T), then the group $\mathrm{FSym}_{\mathrm{ext}}(H)$ is sofic but not residually amenable. The group $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ is used in [BrieusselZheng, 2019, Proposition 4.4] to provide an example of a locally-finite-by-$\mathbb{Z}$ group that does not possess Shalom's property H_{FD}.

Random walks on lampshuffler groups have also been studied in the literature. It is shown in [Yadin, 2009] that the drift function of the simple random walk for the standard generating set on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ is asymptotically equivalent to $n^{3/4}$. In [ErschlerZheng, 2020, Corollary 1.4] it is proved that the Følner function of $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$, $d \geq 1$, is asymptotically equivalent to

n^{n^d}, and the return probability $\mu^{2n}(e)$ of the simple random walk is shown to be asymptotically $\exp\left(-n^{\frac{d}{d+2}}\log^{\frac{2}{d+2}} n\right)$. Given a random walk (G,μ), the problem of realizing any number in $[0, h(G,\mu)]$ as the Furstenberg-entropy of some ergodic (G,μ)-space (called a "full realization") is studied in [HartmanYadin, 2018] for wreath products and lampshuffler groups. It is shown that any non-degenerate random walk with finite entropy on a wreath product $A \wr B$ that induces a Liouville random walk on B admits a full realization [HartmanYadin, 2018, Theorem 1.3]. Similarly, if X is a countable set and $H \subseteq \mathrm{Sym}(X)$ is a finitely generated nilpotent subgroup of (possibly infinitely supported) permutations of X, then for any non-degenerate random walk with finite entropy, the extension $\mathrm{FSym}(X) \rtimes H$ has full realization [HartmanYadin, 2018, Theorem 1.4]. In particular, this holds for the lampshuffler groups $\mathrm{FSym}_{\mathrm{ext}}(H)$ for H finitely generated nilpotent. Lampshuffler groups also appear in [FeldheimSodin, 2022], where the "umpteen operator" is introduced as a representation-theoretic analog of a random Schrödinger operator, and the property of having Lifshitz tails is linked to the decay of the return probability of the simple random walk on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$.

Part I

Unbounded depth on wreath products

Chapter 2

Dead ends on wreath products

This chapter corresponds to the article [Silva, 2023a].

Abstract

For any finite group A and any finitely generated group B, we prove that the corresponding lamplighter group $A \wr B$ admits a standard generating set with unbounded depth, and that if B is abelian then the above is true for every standard generating set. This generalizes the case where $B = \mathbb{Z}$ together with its cyclic generator [Cleary-Taback, 2005a]. When $B = H * K$ is the free product of two finite groups H and K, we characterize which standard generators of the associated lamplighter group have unbounded depth in terms of a geometrical constant related to the Cayley graphs of H and K. In particular, we find differences with the one-dimensional case: the lamplighter group over the free product of two sufficiently large finite cyclic groups has uniformly bounded depth with respect to some standard generating set.

Contents

2.1 Introduction

Let G be a finitely generated group endowed with the word length $\|\cdot\|_S$ associated with a finite symmetric generating set S. By definition, any element $g \in G$ is the endpoint of a geodesic path from e_G to g, and one can ask whether such paths can be extended beyond g, while remaining geodesic. Any element that fails to satisfy the above is called a *dead end* of G with respect to S.

The existence of dead end elements in a group is actually not so rare. For example, $\mathbb{Z}$ with the generating set $\{\pm 2, \pm 3\}$ has 1 and -1 as dead ends of word length 2. However, these are the only ones in this example, and in fact any abelian group always has finitely many dead ends [Lehnert, 2009; Šunić, 2008]. For a virtually abelian group G a slightly weaker condition holds: there exists a constant $M \geq 0$, which depends on the choice of S, such that any element $g \in G$ is at distance at most M from a geodesic path of length $\|g\|_S + 1$ that starts at e_G [Warshall, 2010]. In such a case we say that (G, S) has *uniformly bounded depth*. In addition to virtually abelian groups, this property is satisfied for any choice of S by hyperbolic groups [Bogopolski, 1997; Warshall, 2010] and by groups with two or more ends [Lehnert, 2009]. Intuitively, if (G, S) has uniformly bounded depth, then geodesic paths starting at e_G can be connected to longer ones at the cost of backtracking a constant number of steps.

If (G, S) does not have uniformly bounded depth, we say that it has *unbounded depth*. This means that one can find, for an arbitrary $n \geq 1$, elements $g \in G$ whose n-neighborhood is contained in the ball of radius $\|g\|_S$ centered at e_G. The existence of infinite groups with this property is not evident, and the first example was given by Cleary and Taback [ClearyTaback, 2005a], who showed that the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z} = \langle a, t \mid a^2, [a, t^i a t^{-i}], i \in \mathbb{Z} \rangle$ has unbounded depth with respect to the generating set $\{a, t^{\pm 1}\}$. This result is a consequence of an explicit formula for the word length of elements in $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$, as we explain now. In general, the word length in a wreath product $A \wr B$ with respect to standard generators can be expressed in terms of the word length in A and the length of solutions to the Traveling Salesperson Problem (TSP) on the corresponding Cayley graph of B (as shown by Parry in [Parry, 1992], and explained in Subsection 2.2.4). In the case of Cleary and Taback one has $B = \mathbb{Z}$ with generating set $\{t^{\pm 1}\}$, and hence the exceptionally simple solutions for the TSP on a line provide an explicit formula for the word length in $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$. Similar arguments hold when replacing $\mathbb{Z}$ with a free group of finite rank together with a free generating set (see Section 2.3), but for other groups (or even free groups with other generating sets) this problem is in general computationally harder and one cannot hope for explicit solutions. Notably, the problem of finding the word length of a given element in $A \wr B$ is known to be NP-hard whenever B is a finitely generated abelian group which contains $\mathbb{Z}^2$ [KharlampovichMoghaddam, 2012].

2.1.1 Main results

In this article, we study the depth properties of more general lamplighter groups $A \wr B$ with respect to standard generating sets, where A is fixed to be a non-trivial group with unbounded depth for some generating set S_A (in particular, A can be any non-trivial finite group). Our first result states that standard generators with unbounded depth always exist.

Theorem A (= Theorem 2.4.8)**.** Let (A, S_A) have unbounded depth and B be any finitely generated group. Then there exists a finite generating set S_B of B for which $(A \wr B, S_A \cup S_B)$ has unbounded depth.

Here it is essential that we consider *standard* generating sets for the wreath product $A \wr B$. Indeed, Warshall has proved that the group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ (and many other solvable groups) admit non-standard generating sets with uniformly bounded depth [Warshall, 2008].

We prove Theorem A in Section 2.4 via the study of spanning paths of minimal length of finite subsets of the Cayley graph of B with respect to S_B. We are interested in how this length varies when modifying the endpoint of said paths. Because of this, an important case for us is where balls of B centered at the identity are close to being Hamiltonian-connected (i.e. there is a Hamiltonian path connecting any two vertices), up to repeating a constant number of elements (Definition 2.4.4). The existence of such a Cayley graph for an arbitrary group B is proved in Lemma 2.4.7, and it is a consequence of the fact that the cube of any finite connected graph is Hamiltonian-connected (Lemma 2.2.3).

Another family of graphs that is close to being Hamiltonian-connected are "rectangular grids" in $\mathbb{Z}^2$ (see Subsection 2.4.4 and Lemma 2.4.10 for precise definitions). By showing that there are bijective 1-Lipschitz embeddings of these graphs onto the Cayley graph of any (infinite) abelian group, with the exception of $\mathbb{Z}$ with its cyclic generator, we obtain the following result: whenever B is abelian, *every* choice of S_B gives rise to a standard generating set of $A \wr B$ with unbounded depth (Proposition 2.4.15).

Next, we prove that the claim of Theorem A cannot in general hold for every standard generating set. That is, we find a finitely generated group B together with a finite generating set S_B for which the associated lamplighter group has uniformly bounded depth (Corollary 2.5.1). This example is explained in Subsection 2.5.1, and is obtained through the study of lamplighters over free products of finite groups. In order to formulate our results, we define for a finite group G with a generating set S_G the *Hamiltonian difference* $\mathscr{H}(G, S_G)$, which measures how much shorter a minimal spanning cycle of $\mathrm{Cay}(G, S_G)$ is in comparison with minimal spanning paths from e_G to a non-identity element (Definition 2.5.2). The main result of Section 2.5 is the following.

Theorem B (= Theorem 2.5.3)**.** Let (A, S_A) have unbounded depth, and consider two finite groups H and K with generating sets S_H and S_K, respectively. Then $(A \wr (H * K), S_A \cup S_H \cup S_K)$ has uniformly bounded depth if and only if $\mathscr{H}(H, S_H) + \mathscr{H}(K, S_K) \geq 1$.

In particular, whenever $\mathrm{Cay}(H, S_H)$ and $\mathrm{Cay}(K, S_K)$ are sufficiently long cycles, the lamplighter group $A \wr (H * K)$ has uniformly bounded depth (Corollary 2.5.7). This seems to be the first examples of lamplighter groups with uniformly bounded depth with respect to standard generators, in contrast with the already mentioned example by Warshall of a non-standard generating set of $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ with uniformly bounded depth [Warshall, 2008].

2.1.2 Background and ideas of the proofs

In the remainder of the Introduction we give more background about the study of dead ends in finitely generated groups, and explain the main ideas of the proofs.

The first definition of dead ends in the literature is commonly attributed to Bogopolski [Bogopolski, 1997], and appears in his proof of the fact that two commensurable hyperbolic groups must be bi-Lipschitz equivalent (soon after, this result was proved to hold for non-amenable groups by Whyte [Whyte, 1999] and Nekrashevych [Nekrashevych, 1998], without referencing dead ends in their proofs). Some other contexts where dead ends occur in the study of the geometry of Cayley graphs are mentioned in the following non-exhaustive list.

1. If (G, S) has unbounded depth, then the language of geodesic words with respect to S cannot be regular [Warshall, 2010].
2. Dead ends appear as points of non-negative conjugation curvature, a notion of "medium scale" Ricci curvature for Cayley graphs introduced by Bar-Natan, Duchin and Kropholler [Bar-NatanDuchinKropholler, 2020], and often lead to finding elements of strictly positive conjugation curvature [KrophollerMallery, 2020].
3. Dead ends of arbitrarily large retreat depth are an obstruction to the connectedness of thickened spheres of Cayley graphs, as studied by Brieussel and Gournay [BrieusselGournay, 2018].
4. A zero asymptotic density of dead ends in the balls of the group is used as an assumption by Saito in [Saito, 2010, Section 11.2, Assumption 2. **S**]. It was later remarked by Calegari and Fujiwara that this is quite restrictive, since there are hyperbolic groups with standard generating sets that have a positive density of dead ends (of uniformly bounded depth) [CalegariFujiwara, 2015].

As we remarked above, the first known examples of groups with unbounded depth were provided by Cleary and Taback [ClearyTaback, 2005a] using *wreath products*, which we explain in more detail now. Given two groups A and B, we define their *wreath product* $A \wr B$ as the semidirect product $\bigoplus_B A \rtimes B$, where B acts by translations on the group $\bigoplus_B A$ of finitely supported functions $f : B \to A$. We say that a generating set of $A \wr B$ is *standard* if it is of the form $S_{\text{std}} = S_A \cup S_B$, where S_A and S_B are generating sets of A and B, respectively.

Even though the Cayley graph of a wreath product $A \wr B$ with respect to a standard generating set is not completely understood (descriptions of Cayley graphs of lamplighter groups $\mathbb{Z}/n\mathbb{Z} \wr \mathbb{Z}$ as Diestel-Leader graphs are known for non-standard generators [Woess, 2005]), the word metric can be described in terms of the ones of A and B. As we explain in Subsection 2.2.4, the word length of an element in $(A \wr B, S_{\text{std}})$ can be expressed in terms of the minimal length of paths in $\text{Cay}(B, S_B)$ which start at e_B, visit some finite subset of elements $F \subseteq B$, and finish in some other one $x \in B$. These paths are solutions to the Traveling Salesperson Problem (TSP) in $\text{Cay}(B, S_B)$, and we denote the length of a minimal such path by $\text{TS}(e_B, x, F)$. More precisely, Equation (2.1) says that for $g = (f, x) \in A \wr B$,

$$\|g\|_{S_{\text{std}}} = \sum_{v \in \text{supp}(f)} \|f(v)\|_A + \text{TS}\left(e_B, x, \text{supp}(f)\right).$$

A common way to interpret this formula is to think of $\text{Cay}(B, S_B)$ as a street that has lamps at every vertex, where each lamp can be at a different state for each element of A, so that a group element $(f, b) \in A \wr B$ is given by a lamps configuration f and a position $b \in B$. Then the generators of S_B account for moving through the street, while the generators of S_A change the state of the lamp at the current position. This is the origin of the name "lamplighter group", and we call A the *lamps group* and B the *base group*. When $\text{Cay}(B, S_B)$ is a tree, it is possible to give a simple description of the solutions to the TSP inside the graph, and hence obtain an explicit formula for the word length in $A \wr B$ (Lemma 2.3.2). This is in particular used by Cleary and Taback, who studied lamplighter groups of the form $A \wr \mathbb{Z}$, where A has unbounded depth with respect to some generating set S_A, and $S_{\mathbb{Z}}$ is the cyclic generating set of $\mathbb{Z}$. They proved that the standard generating set $S_{\text{std}} = S_A \cup S_{\mathbb{Z}}$ has unbounded depth, and thus provided the first examples of groups with this property [ClearyTaback, 2005a]. Their arguments strongly rely on the fact that the Cayley graph of the base group is a line, and generalizes to finitely generated free groups (Proposition 2.3.3). On the other hand, for other base groups, or even other generators of $\mathbb{Z}$, the TSP is known to be computationally hard and hence it is not possible to hope for an explicit description of the word length of a wreath product.

Our results concern the existence of dead ends of arbitrarily large depth in wreath products $A \wr B$ over more general base groups B. Note that if the group A has uniformly bounded depth with respect to S_A, then it is straightforward to see that $A \wr B$ also does with respect to S_{std}. Because of this, we concentrate on the case where (A, S_A) has unbounded depth and B is any group with a finite generating set S_B. We use the name *lamplighter group* to refer to any such wreath product.

Our focus on dead ends leads us to study the value of $\text{TS}(e_B, v, F)$ in finite connected subgraphs F of $\text{Cay}(B, S_B)$ that contain large balls centered at the identity e_B. In many cases, we observe that these solutions are actually Hamiltonian paths: they visit each vertex of F exactly once, except possibly for one element when the path is a cycle (see Example 2.4.1). Thanks to the Fuzz Lemma 2.2.7, frequently used by Warshall for studying depth properties of groups [Warshall, 2008; Warshall, 2010; Warshall, 2011], it is enough to estimate the word length up to an additive constant. This together with the above remark on Hamiltonian paths motivates the following definition. A Cayley graph $\text{Cay}(B, S_B)$, where B is a group with a finite generating set S_B, is said to be *quasi-Hamiltonian* if there exists a constant $M \geq 0$ such that for any $n \in \mathbb{N}$, there exists a finite subset $F \subseteq B$ which contains the ball $B_{S_B}(e_B, n)$ and for which $|\text{TS}(e_B, v, F) - |F|| \leq M$, for any $v \in F$. This is, a path of minimal length which starts at e_B, visits all elements of F and finishes at any $v \in F$, while visiting each element of F at most once except for a bounded number of instances.

In Lemma 2.4.5 we prove that if $\text{Cay}(B, S_B)$ is a quasi-Hamiltonian presentation, then the corresponding lamplighter group $(A \wr B, S_{\text{std}})$ has unbounded depth. Then, using results about Hamiltonian-connectedness of finite graphs, we prove that any infinite group admits a quasi-Hamiltonian Cayley graph (Lemma 2.4.7), and hence that any lamplighter group $A \wr B$ admits a standard generating set with unbounded depth (Theorem 2.4.8).

Having proved the existence of at least one quasi-Hamiltonian Cayley graph for any group, one may wonder if in general an arbitrary generating set will have this property. A clear restriction is that if $\text{Cay}(B, S_B)$ is a tree then it cannot be quasi-Hamiltonian, since any path visiting all vertices in a ball is forced to repeat an unbounded amount of them (Example 2.4.3). It turns out that this is the only constraint in the family of abelian groups: we prove that any Cayley graph of a finitely generated abelian group is quasi-Hamiltonian, with the exception of $\text{Cay}(\mathbb{Z}, \{\pm 1\})$ (Proposition 2.4.14). In order to show this result, we use the fact that grid graphs (finite induced subgraphs of $\mathbb{Z}^2$ with canonical generators, with vertex set $[0,n]\times[0,m]$) have spanning paths between any pair of vertices that repeat at most 2 elements. This follows from a characterization of the existence of Hamiltonian paths in grid graphs due to Itai, Papadimitriou and Szwarcfiter [ItaiPapadimitriouSzwarcfiter, 1982]. Then, an inductive argument shows that any Cayley graph of an abelian group, except for $\mathrm{Cay}(\mathbb{Z},\{\pm 1\})$, contains grid graphs as spanning subgraphs of sets containing arbitrarily large finite balls. By combining the above together with the original result of Cleary and Taback, we obtain as a corollary that any lamplighter group $A \wr B$ over an abelian group B has unbounded depth, with respect to every standard generating set (Proposition 2.4.15).

The study of Hamiltonian paths in Cayley graphs has a long history. Lovász Conjecture (for Cayley graphs) asks if any Cayley graph of a finite group has a Hamiltonian cycle and, although it has been verified for various families of groups, it remains far from being solved. We refer to Subsection 2.2.3 for more details and to [LanelPallageRatnayakeThevashaWelihinda, 2019] for a recent survey on the topic. On the other hand, for infinite groups there has been progress in the question of finding Hamiltonian paths or Hamiltonian circles in their Cayley graphs, that is, homeomorphic copies of the interval $[0,1]$ or the circle S^1, respectively, in the Freudenthal compactification of the graph [MiraftabRühmann, 2018]. We emphasize that our definition of quasi-Hamiltonian presentations concerns a slightly different question to the ones above since, despite our interest in infinite groups, we concentrate on paths covering finite (arbitrarily large) subgraphs of an infinite Cayley graph.

Even though Cayley graphs that are trees are not quasi-Hamiltonian, lamplighter groups over them still have unbounded depth with respect to standard generators (Proposition 2.3.3). Hence it is natural to ask whether standard generating sets in lamplighter groups always have unbounded depth. We show that this is not the case, by constructing lamplighter groups which have uniformly bounded depth with respect to standard generators (Corollary 2.5.1). The Cayley graphs of the base groups of the above examples have cut vertices, that prevent them from being quasi-Hamiltonian, but at the same time contain sufficiently long cycles that allow an element to increase its word length by moving the position of the lamplighter. This construction seems to provide the first examples of lamplighter groups with uniformly bounded depth with respect to some standard generating set, in contrast with Warshall's results about non-standard generators with the same property [Warshall, 2008].

The above result is a particular case of our study of lamplighters over a free product of two finite groups (H,S_H) and (K,S_K). In this case, the depth properties of $A\wr(H*K)$ with respect to $S_{\mathrm{std}} = S_A \cup S_H \cup S_K$ are closely related to the solutions of the TSP in the finite graphs $\mathrm{Cay}(H,S_H)$ and $\mathrm{Cay}(K,S_K)$. More precisely, for a group G with a generating set S_G we define and study the *Hamiltonian difference*

$$\mathscr{H}(G,S_G) \coloneqq \max_{g\in G\setminus\{e_G\}} \left\{ \mathrm{TS}\left(e_G, g, G\right) \right\} - \mathrm{TS}\left(e_G, e_G, G\right),$$

where we recall that for any $g \in G$, $\mathrm{TS}\left(e_G, g, G\right)$ denotes the length of a path of minimal length in $\mathrm{Cay}(G,S_G)$ which starts at e_G, finishes at g, and visits all elements of G. When $\mathrm{Cay}(G,S_G)$

is Hamiltonian-connected we have $\mathscr{H}(G, S_G) = -1$, while on the other hand, $\mathscr{H}(G, S_G)$ can take any positive value if $\mathrm{Cay}(G, S_G)$ is chosen to be a sufficiently long cycle. The Hamiltonian difference measures how much shorter minimal spanning cycles are than minimal spanning paths from e_G to a non-identity element inside $\mathrm{Cay}(G, S_G)$.

We prove that the value of $\mathscr{H}(H, S_H) + \mathscr{H}(K, S_K)$ completely characterizes the existence of dead ends of arbitrarily large depth in $(A \wr (H * K), S_{\mathrm{std}})$ (Theorem 2.5.3), and we further detail the case of free products of finite abelian groups (Corollary 2.5.7). Notably, a lamplighter over the free product of two sufficiently long cycles has uniformly bounded depth. Lamplighters over free products of finite groups also provide an interesting contrast to the strict structure of dead end elements of lamplighters over trees (Example 2.5.4).

The organization of the article is as follows. In Section 2.2 we introduce the notation and basic tools we use. We also define wreath products and interpret the word length of an element in standard generating sets through solutions to the TSP in $\mathrm{Cay}(B, S_B)$ (Equation 2.1). Then, in Section 2.3 we discuss lamplighter groups with B finite and with B a free group, which correspond to cases where the solutions to the TSP have a simple structure (with respect to our objectives of studying depth properties). We introduce the quasi-Hamiltonian property in Section 2.4 and study its consequences on the depth properties of lamplighters. Notably, we prove Lemma 2.4.7 about the existence of quasi-Hamiltonian Cayley graphs, and use it to prove Theorem 2.4.8. The section finishes with Propositions 2.4.14 and 2.4.15, regarding lamplighters over abelian groups. Finally, in Section 2.5 we study lamplighters over free products of finite groups, describe explicitly a lamplighter group with uniformly bounded depth for a standard generating set in Subsection 2.5.1, and then prove Theorem 2.5.3.

2.2 Preliminaries

2.2.1 Notation and definitions of graph theory

We start by recalling essential concepts of graph theory and by fixing our notation.

A graph Γ is a pair (V, E), where $V = V(\Gamma)$ and $E = E(\Gamma)$ are the sets of vertices and edges of Γ, that is, E consists of unordered pairs of vertices. We will work with graphs with finite as well as infinite sets of vertices and edges. For the purposes of this paper, these sets will always be countable and graphs will be locally finite, meaning that each vertex forms part of finitely many edges.

A path P in Γ is a sequence of (not necessarily distinct) vertices $P = v_1, v_2, \ldots, v_n \in V$ such that for all $1 \leq i < n$, there is an edge connecting v_i to v_{i+1}, and we say that the length of P is n. If moreover $v_1 = v_n$, we say that P is a cycle.

If Γ is a finite graph, a *spanning path* (resp. *spanning cycle*) P is one that visits each vertex of the Γ. If every vertex is visited a unique time (except for the final one in the case of a cycle), we call P a *Hamiltonian path* (resp. *Hamiltonian cycle*).

Definition 2.2.1. A finite graph is said to be *Hamiltonian* if it possesses a Hamiltonian cycle, and *Hamiltonian-connected* if for any pair of distinct vertices, there is a Hamiltonian path connecting them.

Any Hamiltonian-connected graph is of course Hamiltonian, but the opposite is not true: cycles of length $n \geq 4$ are Hamiltonian but no Hamiltonian-connected. This will be relevant for us in Section 2.5 when studying lamplighter groups over free products of cyclic groups.

A general obstruction to Hamiltonian-connectedness is being bipartite, meaning that the vertex set can be decomposed into two disjoint subsets A and B with no edges between them. In that case a parity argument shows that, if the graph has at least 3 vertices, there cannot be Hamiltonian paths between any two vertices. A bipartite graph is called *Hamiltonian-laceable* if there is a Hamiltonian path between any pair of vertices in different sets of the partition V_Γ. This is our way of saying that Γ has as many Hamiltonian paths as possible, given that it is bipartite.

When proving that a graph is not Hamiltonian-connected, common techniques usually rely on showing that a path visiting all vertices gets trapped at some vertex and must be forced to repeat other ones in order to finish where it is supposed to. This suggests that, if we allow the path to jump a finite bounded distance instead of only moving through adjacent vertices, we may be able to find paths which visit all vertices exactly once, with any starting and finishing point. In order to formalize that intuition, we make the following definition.

Definition 2.2.2. Given a finite connected graph Γ and a positive integer $k \geq 1$, we define the k-th power graph Γ^k of Γ as the graph with $V(\Gamma^k) = V(\Gamma)$ and

$$E(\Gamma^k) := \{uv \mid u, v \in V(\Gamma) \text{ such that } d(u, v) \leq k\}.$$

Commonly, Γ^2 is called the *square* of Γ and Γ^3 the *cube* of Γ.

The following result states that the cube of a finite connected graph is always Hamiltonian-connected. It was proved independently by Sekanina [Sekanina, 1960] and Karaganis [Karaganis, 1968]. This result can be proved by noting that it suffices to show it for a spanning tree of Γ, where an inductive argument can be applied.

Lemma 2.2.3. *Let Γ be any finite connected graph. Then Γ^3 is Hamiltonian-connected.*

Lemma 2.2.3 has been generalized to infinite graphs by Sekanina [Sekanina, 1960], who proved that the third power of any locally finite, 1-ended graph has a spanning ray, and by Heinrich [Heinrich, 1978], who extended this fact to a class of non-locally finite graphs. With respect to Cayley graphs, Georgakopoulos used this result in [Georgakopoulos, 2009] to prove that any finitely generated group G admits a generating set S for which $\mathrm{Cay}(G, S)$ has a Hamiltonian circle. We will use similar ideas in order to prove Theorem 2.4.8.

The conclusion of Lemma 2.2.3 does not hold in general if we replace the cube of the graph by its square [ChartrandLesniakZhang, 2016, Figure 6.14]. However, Fleischner [Fleischner, 1974a; Fleischner, 1974b] proved that if one adds the extra hypothesis that the graph is 2-connected, then its square must have a Hamiltonian cycle. Moreover, Fleischner's result actually implies Hamiltonian-connectedness of the square of 2-connected graphs, as proved by Chartrand, Hobbs, Jung, Kapoor and Nash-Williams [ChartrandHobbsJungKapoorNash-Williams, 1974].

We finish this subsection by giving the formal definition of the direct product of two graphs, which will be useful in Subsection 2.4.4 when discussing Hamiltonian-connected properties of Cayley graphs of infinite abelian groups.

Definition 2.2.4. Let Γ_1, Γ_2 be two graphs. We define their product $\Gamma = \Gamma_1 \times \Gamma_2$ as the graph whose vertex set is $V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2)$ and where two vertices $(u_1, u_2), (v_1, v_2) \in V(\Gamma)$ are connected by an edge if and only if $u_1 = v_1$ and u_2 is connected by an edge in $E(\Gamma_2)$ to v_2, or if $u_2 = v_2$ and u_1 is connected by an edge in $E(\Gamma_1)$ to v_1.

2.2.2 Cayley graphs and word length

Whenever we talk about groups, we assume that they are finitely generated. We denote by (G, S) a group together with a finite (symmetric) generating set S. We use the notation e_G for the identity element of the group G, or simply e if there is no risk of confusion.

The (right, undirected, unlabeled) Cayley graph $\mathrm{Cay}(G, S)$ of G with respect to the generating set S is the graph whose vertices are the elements of G, and where two elements g, g' are connected through an edge if and only if there exists $s \in S \cup S^{-1}$ with $g = g's$. In this context, a natural metric arises in G. Indeed, define for $g, h \in G$,

$$d_S(g,h) \coloneqq \min\left\{n \geq 0 \mid g^{-1}h = s_1 \cdots s_n, \text{ for some } s_1, \ldots, s_n \in S \cup S^{-1}\right\}.$$

That is, $d_S(g, h)$ is the length of a minimal path in $\mathrm{Cay}(G, S)$ connecting g to h. The distance d_S is called the word metric on G associated with the generating set S. It corresponds to the minimum number of generators of S (and their inverses) one needs to multiply to g in order to obtain the element h. Similarly, we define the word length associated with S as

$$\|g\|_S = d_S(e_G, g), \text{ for } g \in G.$$

Given $g \in G$ and $n \geq 0$, we define the ball of radius n centered at g as

$$B(g,n) \coloneqq \{h \in G \mid d_S(h,g) \leq n\}.$$

When there is risk of confusion, we use the notation $B_S(g, n)$ or $B_{(G,S)}(g, n)$ to emphasize the generating set or the group used to define the ball.

In Section 2.4 we will be interested in finding, for each $n \geq 1$, spanning paths of finite subgraphs of $\mathrm{Cay}(G, S)$ containing $B(e_G, n)$, whose length is close to the length of a hypothetical Hamiltonian path, up to a uniform additive error.

2.2.3 Hamiltonian paths on finite Cayley graphs

The problem of finding Hamiltonian cycles on Cayley graphs of finite groups was first proposed by Elvira Rapaport Strasser [Rapaport, 1959], and then by Lovász in 1969 [Bondy-Murty, 1976, Appendix IV. Problem 20]. Lovász conjectured that every finite connected vertex-transitive graph has such one cycle, except for five known counterexamples: the complete graph on 2 vertices, the Petersen graph, the Coxeter graph, and the graphs obtained by replacing in one of the last two graphs each vertex by a triangle. None of these counterexamples are Cayley graphs of groups, and hence the version Lovász conjecture for Cayley graphs of finite groups asks if any such graph with at least 3 elements has a Hamiltonian cycle. Up until now this conjecture remains open, although it has been verified for various families of groups. Surveys on this topic can be found in [CurranGallian, 1996; LanelPallageRatnayakeThevashaWelihinda, 2019; Witte-Gallian, 1984]. Notably, it is a well established fact that any Cayley graph of a finite abelian group with at least 3 elements has a Hamiltonian cycle [Marušič, 1983].

With respect to properties such as Hamiltonian-connectedness or Hamiltonian-laceability, it is quick to find Cayley graphs which have none of these two properties. For example, any cycle of length bigger or equal than 5 of even length is neither Hamiltonian-connected nor Hamiltonian-laceable, and hence such examples are found even within the family of finite cyclic groups. However, in 1981 Chen and Quimpo proved that among finite abelian groups these are the only counterexamples one can find.

Proposition 2.2.5 ([ChenQuimpo, 1981])**.** *Let* Γ *be a Cayley graph of a finite abelian group. Then if* Γ *is not a cycle, either*

1. Γ *is not bipartite and Hamiltonian-connected, or*
2. Γ *is bipartite and Hamiltonian-laceable.*

It is natural to ask if the conclusion of Proposition 2.2.5 holds in general, that is, if any Cayley graph of a finite group G of degree at least 3 is either Hamiltonian-connected, or it is bipartite and Hamiltonian-laceable [DupuisWagon, 2015, Questions 4.1-4.3]. As we saw above, this holds if G is abelian, and it has also been proved using computational methods for groups of order $|G| < 48$ [Witte MorrisWilk, 2020]. Some other particular families of groups have been shown to satisfy this property [Alspach, 2015; AlspachChenMcAvaney, 1996; AlspachQin, 2001; Araki, 2006], but the general case is far from being solved.

2.2.4 Wreath products and lamplighter groups

For A, B groups, we define their *wreath product* $A \wr B$ as the semidirect product $\bigoplus_B A \rtimes B$, where $\bigoplus_B A$ is the group of finitely supported functions $f : B \to A$ endowed with the operation $\oplus$ of componentwise multiplication. We denote by $\mathrm{supp}(f)$ the finite subset of B to which f assigns non-trivial values. Here, the group B acts on the direct sum $\bigoplus_B A$ from the left by translations. That is, for $f : B \to A$, and any $b \in B$ we have

$$(b \cdot f)(x) = f(b^{-1}x), \ x \in B.$$

Elements of $A \wr B$ can be expressed as a tuple (f, b), where $f : B \to A$ is a finitely supported function and $b \in B$, and the product between two such elements elements (f, b), $(f', b') \in A \wr B$ is given by

$$(f, b) \cdot (f', b') = (f \oplus (b \cdot f'), bb').$$

There is a natural embedding of B into $A \wr B$ via the mapping

$$\begin{aligned} B &\to A \wr B \\ b &\mapsto (\mathbb{1}, b), \end{aligned}$$

where $\mathbb{1}(x) = e_A$ for any $x \in B$. Similarly, we can embed A into $A \wr B$ via the mapping

$$\begin{aligned} A &\to A \wr B \\ a &\mapsto (\delta^a_{e_B}, e_B), \end{aligned}$$

where $\delta^a_{e_B}(e_B) = a$ and $\delta^a_{e_B}(x) = e_A$ for any $x \neq e_B$.

In particular, if we consider finite symmetric generating sets S_A and S_B of A and B, respectively, their copies inside $A \wr B$ through the above embeddings generate the entire group $A \wr B$. We call $S_{\text{std}} \coloneqq S_A \cup S_B$ the *standard generating set* for $A \wr B$ associated with the generators S_A and S_B.

In order to understand how the word length of an element with respect to S_{std} looks like, consider $g = (f, x) \in A \wr B$ and write it as a product of generators of S_{std},

$$g = a_0 b_1 a_1 b_2 a_2 \cdots b_m a_m,$$

with $m \geq 0$, $a_0, \ldots, a_m \in A$, and $b_1, \ldots, b_m \in B$. In particular, it holds that

$$f = a_0 (b_1 \cdot a_1)(b_1 b_2 \cdot a_2) \cdots (b_1 b_2 \cdots b_m \cdot a_m),$$

so that $\operatorname{supp}(f) \subseteq \{e_B, b_1, b_1 b_2, \ldots, b_1 b_2 \cdots b_m\}$, and $x = b_1 b_2 \cdots b_m$. This factorization of g into generators of S_{std} can be interpreted as a path in the Cayley graph $\operatorname{Cay}(B, S_B)$ which begins at e_B, visits all vertices in $\operatorname{supp}(f)$ while generating the appropriate group element for f in each one, and ends at $b_1 b_2 \cdots b_m$. This is the reason behind the name "lamplighter group": one can think of the Cayley graph $\operatorname{Cay}(B, S_B)$ as a street with lamps at every vertex, each of which can be in a different state given by an element of A. Then a word in S_{std} evaluating to an element $g = (f, x) \in A \wr B$ corresponds to a path from the origin e_B to x, which passes through all vertices of $\operatorname{supp}(f)$ and at each one of them uses the generators of S_A in order to reach the value of f there. We refer to x as the *position of the lamplighter*, and to f as the *lamps configuration*.

The above discussion shows that the word length of an element $g = (f, x) \in A \wr B$ with respect to S_{std} can be expressed as

$$\|g\|_{S_{\text{std}}} = \sum_{y \in \operatorname{supp}(f)} \|f(y)\|_{S_A} + \operatorname{TS}(e_B, x, f), \tag{2.1}$$

where $\operatorname{TS}(b, b', f)$ corresponds to the length of a path of minimal length in $\operatorname{Cay}(B, S_B)$ which starts at b, finishes at b' and visits all vertices of $\operatorname{supp}(f)$. The notation TS stands for the interpretation of said walk as a solution to the Traveling Salesperson Problem (TSP). We also write $\operatorname{TS}(b, b', F)$ to denote a walk of minimal length in $\operatorname{Cay}(B, S_B)$ starting at b, finishing at

b' and visiting all vertices from a finite subset $F \subseteq B$.

Equation (2.1) has been widely used to study wreath products. We already mentioned in the Introduction that Cleary and Taback used it to study depth properties of lamplighter groups $A \wr \mathbb{Z}$ [ClearyTaback, 2005a], but it has also been used to study other metric properties of wreath products, as for example by Parry in order to study rationality and algebraicity of growth series [Parry, 1992], by Davis and Olshanskii to study distortion of subgroups of some wreath products [DavisOlshanskii, 2011], among many others.

2.2.5 Dead ends on groups

Fix (G, S) a group together with a finite generating set. We say that an element $g \in G$ is a *dead end* if for any $s \in S \cup S^{-1}$, $\|gs\|_S \leq \|g\|_S$, and we define its *depth* with respect to S as the maximal number $n \geq 1$ such that for any choice of generators $s_1, \dots, s_k \in S \cup S^{-1}$, $k \leq n$, we have $\|gs_1 \cdots s_k\|_S \leq \|g\|_S$. That is, the depth of a dead end is the maximal number n such that multiplying by at most n generators does not increase the word length of g. Another way of saying this is that g maximizes the function $\| \cdot \|_S$ on its n-neighborhood.

Although their origin might be older, the first definition for dead ends is commonly attributed to Bogopolski in 1997 [Bogopolski, 1997], who used it while proving the fact that two commensurable hyperbolic groups are in fact bi-Lipschitz equivalent. Soon after, this property was shown to hold for any non-amenable groups by Whyte [Whyte, 1999] and Nekrashevych [Nekrashevych, 1998] without using the notion of dead ends.

Ideas related to dead ends had already appeared before 1997 in the literature, as for example by Champetier in [Champetier, 1995, Lemme 4.19] where it is proved that group presentations $G = \langle S \mid R \rangle$ satisfying the $C'(1/6)$ small cancellation condition (see [LyndonSchupp, 2001, Chapter V.2] for a definition) satisfy the following property: for any $g \in G$, the set $\{s \in S \cup S^{-1} \mid \|gs\|_S \leq \|g\|\}$ has at most two elements. This is also discussed by de la Harpe in [Harpe, 2000, Chapter IV.A. 13, 14]. In the latter, dead ends are introduced as an obstruction for the *extension property for geodesic segments* of a graph. With respect to more recent literature, dead ends are discussed in the chapters of some books on Geometric Group Theory, as in [BonanomeDeanDean, 2018, Subsections 1.8.5, 2.6.4 & 4.7.2] and in [ClayMargalit, 2017, Chapters 12, 15 & 16].

The depth of a dead end $g \in G$ with respect to a finite generating set S can be interpreted as the distance in $\mathrm{Cay}(G, S)$ between g and the complement of the ball $B_S(e_G, \|g\|_S)$, minus 1. Note that if G is an infinite group, then the depth of any element g has as an upper bound $2\|g\|_S$ (since any infinite finitely generated group contains an infinite geodesic ray). On the other hand, finite groups always have elements of infinite depth: those that maximize the word metric $\| \cdot \|_S$. However, although the depth of each element must be finite, it can be possible that G contains dead ends of arbitrarily large depth.

Definition 2.2.6. Let G be a finitely generated group and S a finite generating set. We say that G has *unbounded depth* if for any $n \in \mathbb{N}$ there exists a dead end $g \in G$ of depth at least n. Otherwise, we say that G has *uniformly bounded depth with respect to S*.

We again emphasize the dependence of these definitions on the choice of the generating set

S. It may happen that a group G has unbounded depth with respect to a generating set S, and uniformly bounded depth with respect to another generating set S'. In fact, Šunić proved in [Šunić, 2008] that any group admits a generating set with dead ends. In the same paper, he proved that $\mathbb{Z}$ always has finitely many dead ends, a fact that would later be proved to hold for any finitely generated abelian group by Lehnert [Lehnert, 2009]. In particular, there are no generating sets with unbounded depth among such groups. Other families of groups which have uniformly bounded depth with respect to any generating set are hyperbolic groups [Bogopolski, 1997], and more generally any group which has a regular language of geodesics for any generating set [Warshall, 2010], and groups with more than one end [Lehnert, 2009]. On the other hand, examples of group with unbounded depth are notably the lamplighter group over the line $\mathbb{Z}/2\mathbb{Z}\wr\mathbb{Z}$ [ClearyElderTaback, 2006; ClearyTaback, 2005a; ClearyTaback, 2005b], and Houghton's group H_2 [Lehnert, 2009]. In general, having unbounded depth is not a group invariant, since there exist groups (even finitely presented ones) which have unbounded depth for one generating set and uniformly bounded depth for another one [ClearyRiley, 2006; RileyWarshall, 2006]. A remarkable exception is the discrete Heisenberg group, which has unbounded depth with respect to any finite generating set, as shown by Warshall [Warshall, 2010; Warshall, 2011].

In order to prove the existence of dead ends in a Cayley graph, it is not necessary to find them explicitly: it suffices to show the existence of elements which increase their word length by a bounded amount when multiplying by a large number of generators. The following lemma formalizes this, and it has been widely used to show the existence of dead ends of arbitrary depth by Warshall [Warshall, 2008; Warshall, 2010; Warshall, 2011].

Lemma 2.2.7 (Fuzz Lemma)**.** *Let X be a metric space, and $f : X \to \mathbb{Z}$ a function. Suppose there exists $M > 0$ such that for some $x \in X$ and $r \in \mathbb{N}^+$ we have*

$$f(x') \leq f(x) + M, \quad \text{for all } x' \in B(x, r).$$

Then there exists some $x_0 \in X$ such that f attains a maximum on $B(x_0, r/M)$ at x_0.

To finish this section we introduce a slightly different notion of depth for dead ends, concerned with how much actual backtracking is needed in order to reach elements of bigger word length.

Definition 2.2.8. Given a dead end $g \in G$ with respect to a generating set S, we say that g has *retreat depth* (or *strong depth*) k if k is the minimal number such that there exists a geodesic from g to an element of $B_S(e_G, \|g\|_S + 1)$ which does not pass through $B_S(e_G, \|g\|_S - k - 1)$.

That is, the retreat depth of g measures how many steps back in $\mathrm{Cay}(G, S)$ from g one needs to take in order to eventually reach a bigger sphere. The retreat depth of an element is bounded above by its depth, but it may be the case that elements of arbitrarily large depth have uniformly bounded retreat depth. Indeed, this is the case for the discrete Heisenberg group: Warshall proved that it has unbounded depth and at the same time uniformly bounded retreat depth, for any generating set [Warshall, 2011]. On the other hand, the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ and Houghton's group $\mathcal{H}_2$ have unbounded retreat depth for standard generating sets [Lehnert, 2009].

Throughout this article, we work with wreath products $A \wr B$ under the assumption that A has an associated finite generating set S_A with unbounded depth, and call it the *lamps group* of the wreath product $A \wr B$. This generalizes the case of finite lamps groups, since finite groups always have elements of infinite depth. We also say that B is the *base group* of $A \wr B$ or that $A \wr B$ is a lamplighter group over B with lamp groups A.

2.3 Lamplighter groups over finite groups and over free groups

This section concerns two particular cases of base groups for which there is a detailed description of the solutions of the TSP, with respect to our purposes of studying depth on lamplighter groups. We show that when the base group is finite, the depth properties are dominated by the lamps group, while on the other hand when the base group is a free group with free generating set (so that the corresponding Cayley graph is a tree) we are able to give a precise characterization of dead end elements using the same arguments of [ClearyTaback, 2005a].

In particular, it follows from Proposition 2.3.3 that dead end elements of lamplighters over trees must necessarily have the position of the lamplighter at the identity element. This restriction does not hold in general for other base groups, and through our study of lamplighters over free products of finite groups in Section 2.5, we find a lamplighter group with a standard generating set whose dead end elements can have the position of the lamplighter at an arbitrarily large distance from the identity element (Example 2.5.4).

2.3.1 Lamplighters over finite groups

We start by proving that when B is finite, the lamplighter group $(A \wr B, S_{\text{std}})$ has the same depth properties as (A, S_A).

Proposition 2.3.1. *Consider (A, S_A) any finitely generated group and (B, S_B) a finite group. Then $(A \wr B, S_{\text{std}})$ has unbounded depth if and only if (A, S_A) does.*

Proof. Suppose first that (A, S_A) has uniformly bounded depth, so that for some $k \geq 1$ and any $a \in A$, there exist $\alpha_1, \ldots, \alpha_k \in S_A \cup S_A^{-1}$ so that $\|a\alpha_1 \cdots \alpha_k\|_{S_A} \geq \|a\|_{S_A} + 1$.

For any element $g = (f, x) \in A \wr B$, Equation (2.1) tells us that the word length of g is

$$\|g\|_{S_{\text{std}}} = \|f\|_{S_A} + \mathrm{TS}\,(e_B, x, f)\,.$$

For $a = f(x)$, find $\alpha_1, \ldots, \alpha_k \in S_A \cup S_A^{-1}$ so that$\|a\alpha_1 \cdots \alpha_k\|_{S_A} \geq \|a\|_{S_A} + 1$. Then

$$g\alpha_1 \cdots \alpha_k = (f \cdot x\alpha_1 \cdots \alpha_k x^{-1}, x),$$

and hence

$$\begin{aligned} \|g\alpha_1 \cdots \alpha_k\|_{S_{\text{std}}} &= \|f \cdot x\alpha_1 \cdots \alpha_k x^{-1}\|_{S_A} + \mathrm{TS}\left(e_B, x, f \cdot x\alpha_1 \cdots \alpha_k x^{-1}\right) \\ &\geq \|f \cdot x\alpha_1 \cdots \alpha_k x^{-1}\|_{S_A} + \mathrm{TS}\,(e_B, x, f) \\ &\geq \|f\|_{S_A} + 1 + \mathrm{TS}\,(e_B, x, f) \end{aligned}$$

$$= \|g\|_{S_{\mathrm{std}}} + 1,$$

where the second inequality comes from the fact that the support of f is contained in that of f with the state of the lamp at position x modified. This shows that g has depth at most k.

Now suppose that (A, S_A) has unbounded depth, and choose $a \in (A, S_A)$ of depth at least n. Choose $x \in B$ that maximizes the value of $\mathrm{TS}\,(e_B, x, B)$, which exists since we assume B to be finite. Then it is straightforward to prove that the element (f, x), where $f(y) = a$ for all $y \in B$, is a dead end of depth at least n, by using Equation (2.1). Indeed, thanks to our choice of x, the word length of (f, x) can only be increased through the term associated with the lamps configuration, for which at least n generators of S_A are needed.

Note that here it is essential that B is finite, so that f defined as above does indeed give a finitely supported function over B. □

For the rest of the article, we concentrate on lamplighter groups over infinite base groups B.

2.3.2 Lamplighters over free groups

Computing the word length of an arbitrary element using Equation (2.1) is not easy in general since it involves solving the TSP, known to be a computationally hard problem. However, an exceptionally simple formula can be given for $F(S)$ a free group with a free finite generating set S. In order to do so, we need to introduce some notation. Given $u, v \in F(S)$, denote by $[u, v]$ the set of edges of the unique shortest path in $\mathrm{Cay}(F(S), S)$ joining u to v, so that $d_S(u, v) = |[u, v]|$. Similarly, for $H \subseteq F(S)$ finite, denote $[u, H] = \bigcup_{h \in H} [u, h]$.

Lemma 2.3.2 ([BaudierMotakisSchlumprechtZsák, 2021, Theorem 3.1],[ClearyTaback, 2005a])**.** *Let S be a finite set and $F(S)$ the free group over S. Then for any $u, v \in F(S)$ and any finite subset $H \subseteq F(S)$ we have*

$$\mathrm{TS}\,(u, v, H) = 2|[u, H] \backslash [u, v]| + |[u, v]|.$$

In particular, for any element $g = (f, x) \in A \wr F(S)$, Equation (2.1) takes the form

$$\|g\|_{S_{\mathrm{std}}} = \sum_{y \in \mathrm{supp}(f)} \|f(y)\|_{S_A} + 2|[e_B, \mathrm{supp}(f)] \backslash [e_B, x]| + \|x\|_S. \tag{2.2}$$

By using Equation (2.2) we can generalize the results about dead ends of lamplighter groups over the line from [ClearyTaback, 2005a] and actually give a characterization of such elements.

Proposition 2.3.3. *Consider the free group $F(S)$ over a finite set S, and a finitely generated group (A, S_A). Then $g = (f, x) \in (A \wr F(S), S_{\mathrm{std}})$ is a dead end if and only if*

1. *$f(x) \in (A, S_A)$ is a dead end,*
2. *$[e_{F(S)}, \mathrm{supp}(f)]$ contains all edges $[x, xs]$, for $s \in S^{\pm 1}$, and*
3. *$x = e_{F(S)}$.*

Moreover, if (A, S_A) has unbounded (retreat) depth, then $(A \wr F(S), S_{\mathrm{std}})$ also does.

Proof. Suppose first that the three conditions hold. As $x = e_{F(S)}$, Equation (2.2) says that

$$\|g\|_{S_{\text{std}}} = \sum_{y \in \text{supp}(f)} \|f(y)\|_{S_A} + 2|[e_{F(S)}, \text{supp}(f)]|.$$

As $f(x)$ is a dead end of (A, S_A), multiplying by a generator in S_A does not increase word length. On the other hand, multiplying by a generator $s \in S$ gives

$$\begin{aligned}
\|gs\|_{S_{\text{std}}} &= \sum_{y \in \text{supp}(f)} \|f(y)\|_{S_A} + 2|[e_{F(S)}, \text{supp}(f)] \backslash [e_B, s]| + |[e_B, s]| \\
&= \sum_{y \in \text{supp}(f)} \|f(y)\|_{S_A} + 2|[e_{F(S)}, \text{supp}(f)] \backslash [e_B, s]| + 1 \\
&= \sum_{y \in \text{supp}(f)} \|f(y)\|_{S_A} + 2\left(|[e_{F(S)}, \text{supp}(f)]| - 1\right) + 1 \\
&= \|g\|_{S_{\text{std}}} - 1,
\end{aligned}$$

where we used Condition (2) in the penultimate equality. This proves that g is a dead end.

Now let us suppose that $g = (f, x) \in (A \wr F(S), S_{\text{std}})$ is a dead end. Clearly Condition (1) must hold, since otherwise multiplying by a generator in S_A would increase word length. Now consider Condition (2). If there exists $s \in S^{\pm 1}$ such that the edge $[x, xs]$ is not contained in $[e_{F(S)}, \text{supp}(f)]$, then

$$\begin{aligned}
\|gs\|_{S_{\text{std}}} &= \sum_{y \in \text{supp}(f)} \|f(y)\|_{S_A} + 2|[e_{F(S)}, \text{supp}(f)] \backslash [e_B, xs]| + |[e_B, xs]| \\
&= \sum_{y \in \text{supp}(f)} \|f(y)\|_{S_A} + 2|[e_{F(S)}, \text{supp}(f)] \backslash [e_B, x]| + |[e_B, x]| + 1 \\
&= \|g\|_{S_{\text{std}}} + 1,
\end{aligned}$$

which contradicts the fact that g is a dead end. Hence Condition (2) holds.

Similarly, suppose that $x \neq e_{F(S)}$ and choose $s \in S^{\pm 1}$ such that $\|xs\|_S = \|x\|_S - 1$. Then

$$\begin{aligned}
\|g\|_{S_{\text{std}}} &= \sum_{y \in \text{supp}(f)} \|f(y)\|_{S_A} + 2|[e_{F(S)}, \text{supp}(f)] \backslash [e_B, x]| + |[e_B, x]| \\
&= \sum_{y \in \text{supp}(f)} \|f(y)\|_{S_A} + 2\left(|[e_{F(S)}, \text{supp}(f)] \backslash [e_B, xs]| - 1\right) + |[e_B, xs]| + 1 \\
&= \|gs\|_{S_{\text{std}}} - 1.
\end{aligned}$$

In other words, $\|gs\|_{S_{\text{std}}} = \|g\|_{S_{\text{std}}} + 1$ and we again contradict that g is a dead end.

Now let us prove the second part of the proposition. If (A, S_A) has unbounded (retreat) depth, fix for $n \geq 1$ an element $a \in (A, S_A)$ of (retreat) depth n. Define the element $g = (f, e_{F(S)}) \in A \wr F(S)$, where $f(x) = a$ if $\|x\|_S \leq n$ and $f(x) = e_A$ otherwise. Then very similar arguments to the ones given above show that for any $s_1, \ldots, s_{n-1} \in S^{\pm 1}$,

$$\|gs_1 \ldots s_{n-1}\|_{S_{\text{std}}} = \|g\|_{S_{\text{std}}} - (n-1).$$

This implies that g will be a dead end of (retreat) depth at least $n-1$ with respect to S_{std}. □

2.4 The quasi-Hamiltonian property

Equation (2.1) tells us that in order to study word length of $(A \wr B, S_{\text{std}})$, we need to understand (at least partially) the solutions to the TSP in $\text{Cay}(B, S_B)$. This problem is in general NP-hard, and hence we cannot hope to have a precise description of all solutions unless we are in very particular families of graphs (such as trees, which were studied in the previous section).

However, our focus on depth of lamplighter groups brings our attention onto very particular instances of the TSP. By looking at the structure of dead ends of lamplighters over trees given by Proposition 2.3.3, a naive approach to finding dead end elements in $A \wr B$ is to consider configurations of the form $g = (f, e_B) \in A \wr B$, where f is a lamps configuration with support on a ball of radius n centered at e_B. Hence, it is relevant for us to study the solutions to the TSP starting at e_B and visiting all vertices in a big ball around the identity.

In Subsection 2.4.1 we show some examples of solutions to the TSP in some Cayley graphs, which serve as motivation for the rest of this section. Then in Subsection 2.4.2 we define the quasi-Hamiltonian property for a graph and show that any group admits a quasi-Hamiltonian Cayley graph. As a consequence, any lamplighter group has a standard generating set with unbounded depth. Finally, in Subsection 2.4.4 we prove that any Cayley graph of an abelian group, except for $\text{Cay}(\mathbb{Z}, \{\pm 1\})$ is quasi-Hamiltonian. As a corollary, all standard generating sets of lamplighters over abelian groups have unbounded depth.

2.4.1 Examples: solutions to the TSP inside some Cayley graphs

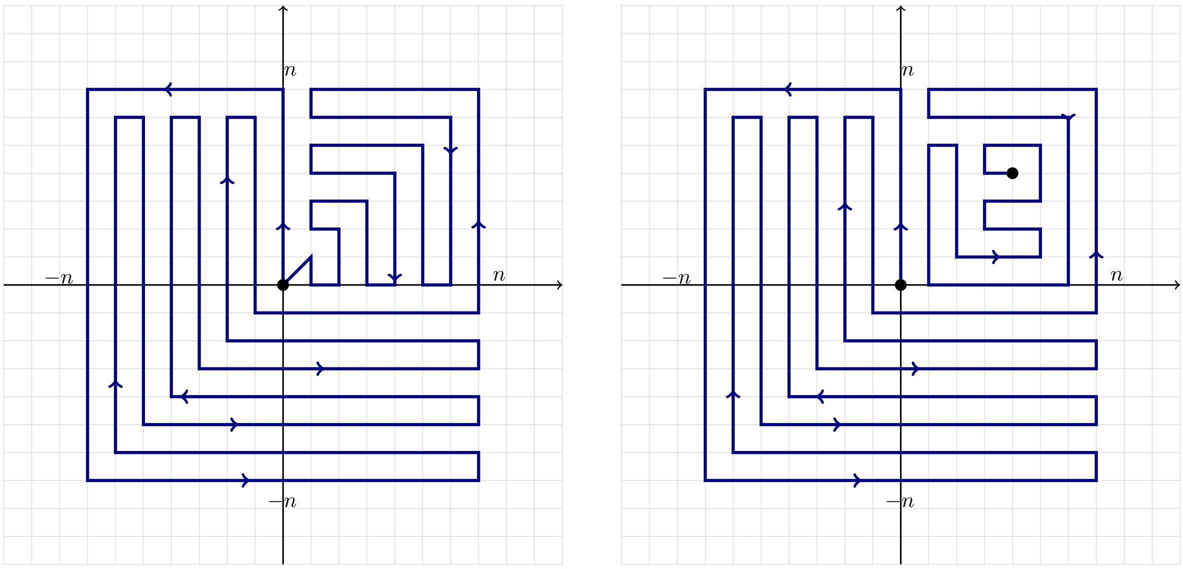
Example 2.4.1. Consider the group $\mathbb{Z}^2$ together with its canonical basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{Z}^2$, and define the *king's moves* generating set $S_{\text{king}} = \{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \mathbf{e}_1 \pm \mathbf{e}_2, -\mathbf{e}_1 \pm \mathbf{e}_2\}$. Then balls of $\text{Cay}(\mathbb{Z}^2, S_{\text{king}})$ have the shape of squares, and it can be proved that the induced subgraphs $B_n \coloneqq B_{S_{\text{king}}}(\mathbf{0}, n)$ are Hamiltonian-connected for any $n \geq 1$. Hence for any $x \in B_n$ we have

$$\text{TS}(\mathbf{0}, x, B_n) = |B_n| + \delta_{x,\mathbf{0}},$$

where $\delta_{\mathbf{0},\mathbf{0}} = 1$ and $\delta_{x,\mathbf{0}} = 0$ otherwise. Indeed, the shortest path from $\mathbf{0}$ to x that covers B_n visits each vertex exactly once, except possibly $\mathbf{0}$ which is visited twice if $x = \mathbf{0}$. Two such solutions are illustrated in Figure 2.1.

Example 2.4.2. Now we consider $\mathbb{Z}$ with a generating set distinct from the cyclic one. Let $S = \{\pm 1, \pm 2\} \subseteq \mathbb{Z}$. The Cayley graph $\text{Cay}(\mathbb{Z}, S)$ is illustrated in Figure 2.2. The ball B_n of radius n centered at 0 is not Hamiltonian-connected, since a path going from 0 to 1 and visiting all vertices of B_n must visit one vertex twice. However, it does hold that

$$|B_n| \leq \text{TS}(0, x, B_n) \leq |B_n| + 1,$$



(a) Ending at $p = (0, 0)$. (b) Ending at $\mathbf{p} = (4, 4)$.

Figure 2.1 – Paths visiting all vertices in the square $[-n, n]^2$, starting at $(0, 0)$ and finishing inside the square.

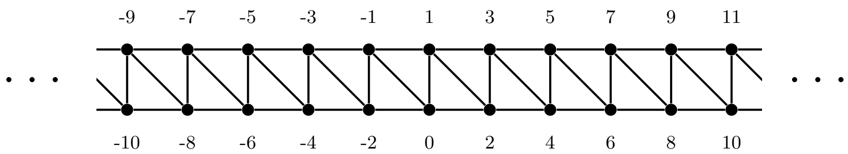


Figure 2.2 – The Cayley graph $\mathrm{Cay}(\mathbb{Z}, \{\pm 1, \pm 2\})$.

for any $x \in B_n$. Hence, even though the graph is not Hamiltonian-connected, solutions to the TSP starting at 0 and visiting all of B_n differ from a hypothetical Hamiltonian path only by a uniform additive constant.

Example 2.4.3. Now consider a finite non-empty set S and the free group $F(S)$. In this case the Cayley graph $\mathrm{Cay}(F(S), S)$ is a tree and for any $x \in B_n := B_S(e_{F(S)}, n)$,

$$\mathrm{TS}\left(e_{F(S)}, x, B_n\right) \geq n + |B_n|.$$

Indeed, a path visiting all vertices in the ball B_n must pass through 0 at least twice, and as $\mathrm{Cay}(F(S), S)$ is a tree this implies that it traverses a geodesic from 0 to an element of word length n twice.

In the first two examples, minimal spanning paths of B_n repeat a constant number of vertices, while on the third one it is necessary to repeat an unbounded number of them. On what follows we study these different behaviors and their consequences of depth properties of lamplighter groups.

2.4.2 The quasi-Hamiltonian property and unbounded depth of lamplighters

We start by defining the property of $\mathrm{Cay}(B, S_B)$ that will be a sufficient condition for the existence of dead ends of unbounded depth in $A \wr B$, and which we believe to be of interest on its own.

Definition 2.4.4 (Quasi-Hamiltonian property)**.** Let Γ be an infinite, connected and locally finite graph, and fix a vertex $o \in V$. Denote by B_n the ball of Γ centered at o of radius n. We say that (Γ, o) has the *quasi-Hamiltonian property* or that it is *quasi-Hamiltonian* if there exists a family $\mathcal{F}$ of connected (induced) finite subgraphs of Γ such that

1. $o \in F$ for every $F \in \mathcal{F}$,
2. for any $n \geq 1$, there exists $F \in \mathcal{F}$ such that $B_n \subseteq F$, and
3. there exists a constant $M \geq 0$ such that for any $F \in \mathcal{F}$ and $x \in F$,

$$\mathrm{TS}\,(o, x, F) \leq |F| + M.$$

We concentrate on the case where $\Gamma = \mathrm{Cay}(B, S_B)$, and $o = e_B$.

Note that Definition 2.4.4 implies that for any $F \in \mathcal{F}$ and $x \in F$,

$$|F| \leq \mathrm{TS}\,(e_B, x, F) \leq |F| + M,$$

so that the lengths of optimal paths are at bounded distance from those of hypothetical Hamiltonian paths.

Lemma 2.4.5. *If* $\mathrm{Cay}(B, S_B)$ *is quasi-Hamiltonian, then for every group* (A, S_A) *of unbounded depth the corresponding lamplighter group* $(A \wr B, S_{\mathrm{std}})$ *has unbounded depth.*

Proof. Since $\mathrm{Cay}(B, S_B)$ has the quasi-Hamiltonian property, there exists $M \geq 0$ such that for any $n \geq 1$ we can find $F \subseteq B$ a connected subgraph with $B_{S_B}(e_B, n) \subseteq F$ and with

$$|F| \leq \mathrm{TS}\,(e_B, p, F) \leq |F| + M,$$

for any $p \in F$.

Fix a dead end $a \in A$ of depth at least n with respect to S_A, and consider the configuration $g = (f, e_B) \in A \wr B$, where $f(x) = a$ if $x \in F$ and $f(x) = e_A$ otherwise. We see that

$$\|g\|_{S_{\mathrm{std}}} \geq \sum_{a \in F} \|a\|_{S_A} + |F|,$$

since in order to light all lamps at vertices of F it is mandatory to visit each of these elements at least once.

Now consider any element $h \in B_{S_{\mathrm{std}}}(e, n)$, and note that the element gh corresponds to a new lamplighter configuration where some of the lamps at positions of F may have changed (by at most n generators), and the new position of the lamplighter is some element $p \in B_{S_B}(e_B, n) \subseteq F$. Such an element can be constructed using the generators of S_{std} by following a spanning path of F starting at e_B and finishing at p, while generating the states of the lamps at configurations of F. We see hence that

$$\|gh\|_{S_{\text{std}}} \leq \sum_{a \in F} \|a\|_{S_A} + |F| + M \leq \|g\|_{S_{\text{std}}} + M.$$

This inequality together with the Fuzz Lemma 2.2.7 prove the existence of a dead end of depth at least n/M. As M does not depend on n, we conclude that $(A \wr B, S_{\text{std}})$ has dead ends of unbounded depth. □

Remark 2.4.6. Suppose we have two generating sets S, S' of B, with $S \subseteq S'$. By noting that $\text{Cay}(B, S')$ can be obtained from $\text{Cay}(B, S)$ by adding a finite number of extra edges at each vertex, we see that if (B, S) has the quasi-Hamiltonian property then so does (B, S').

More generally, if Γ is any subgraph obtained from $\text{Cay}(B, S)$ by removing edges (but not vertices) and Γ has the quasi-Hamiltonian property, then so does $\text{Cay}(B, S)$.

2.4.3 Existence of standard generating sets with unbounded depth

Now we prove that any infinite group admits a quasi-Hamiltonian Cayley graph. In order to do so, we use Lemma 2.2.3, which tells us that the cube of any connected finite graph is Hamiltonian-connected. This result has been used in a similar manner by Georgakopoulos in order to prove the existence of generating sets with Hamiltonian circles in Cayley graphs on infinite groups [Georgakopoulos, 2009], and by Khukhro [Khukhro, 2023] and Ostrovskii and Rosenthal [OstrovskiiRosenthal, 2015] in order to show that non virtually free groups admit Cayley graphs which have any finite graph as a minor.

Lemma 2.4.7 (Existence of a quasi-Hamiltonian Cayley graphs)**.** *Any finitely generated group B admits a quasi-Hamiltonian Cayley graph.*

Proof. Start with any symmetric finite generating set S for B, and for an arbitrary $n \geq 1$ consider $F = B_S(e_B, 3n)$. Thanks to Lemma 2.2.3, the cube of the induced graph by F in $\text{Cay}(B, S)$ is Hamiltonian-connected, and hence F is a Hamiltonian-connected subset of $\text{Cay}(B, S \cup S^2 \cup S^3)$.

Denote this new generating set $S_B := S \cup S^2 \cup S^3$. By definition, $F = B_{S_B}(e_B, n)$ is a Hamiltonian-connected subgraph.

As n was arbitrary, this proves that $\text{Cay}(B, S_B)$ has the quasi-Hamiltonian property. □

Theorem 2.4.8. *Let A be a finitely generated group with unbounded depth for some finite generating set, and B be any finitely generated group. Then there exists a standard generating set of $A \wr B$ with unbounded depth.*

Proof. Thanks to Lemma 2.4.5, the result follows from the existence of a quasi-Hamiltonian presentation given by Lemma 2.4.7. □

Example 2.4.9. Let S be a finite set with at least two elements, and consider the free group $F(S)$. It follows from Example 2.4.3 that $\text{Cay}(F(S), S)$ does not have the quasi-Hamiltonian property, and from Lemma 2.4.7 that $\text{Cay}(F(S), S \cup S^2 \cup S^3)$ does. It is natural to ask about what happens for $\mathrm{Cay}(F(S), S \cup S^2)$, and the answer is that it does not have the quasi-Hamiltonian property.

Indeed, let H be a subset of $F(S)$ that contains a ball of radius n, and let γ be a spanning cycle of H of length $\mathrm{TS}\left(e_{F(S)}, e_{F(S)}, H\right)$. Doing a case by case analysis, it is possible to see that for any element $v \in F(S)$ with $\|v\|_S \leq n$, the path γ must repeat at least one vertex in its 1-neighborhood. This means that $\mathrm{TS}\left(e_{F(S)}, e_{F(S)}, H\right) - |H| \xrightarrow[n\to+\infty]{} +\infty$, and so $(F(S), S \cup S^2)$ is not a quasi-Hamiltonian presentation.

For this example to work, it is essential that $|S| \geq 2$. As we showed in Example 2.4.2, $\mathrm{Cay}(\mathbb{Z}, \{1, 2\})$ does have the quasi-Hamiltonian property.

2.4.4 Abelian groups

In this subsection, we prove that any Cayley graph of a finitely generated abelian group different from $\mathrm{Cay}(\mathbb{Z}, \{\pm 1\})$ is quasi-Hamiltonian. Since the lamplighter group over $\mathrm{Cay}(\mathbb{Z}, \{\pm 1\})$ is covered by Cleary and Taback's original example in [ClearyTaback, 2005a], we conclude that any lamplighter group $A \wr B$ over an abelian base group B has unbounded depth, with respect to every standard generating set $S_{\mathrm{std}} = S_A \cup S_B$ (as long as the lamps group (A, S_A) has unbounded depth).

We begin by sketching the proof of the fact that every Cayley graph of an abelian group other than $\mathrm{Cay}(\mathbb{Z}, \pm 1)$ has the quasi-Hamiltonian property. Our starting point are "grid graphs", that is, graphs whose vertex set is

$$\{1, \ldots, n\} \times \{1, \ldots, m\}, \text{ for } n, m \geq 2,$$

and where edges connect vertices of the form (i, j) with (k, l) if and only if $|i - k| + |j - l| = 1$, for $1 \leq i, k \leq n$ and $1 \leq j, l \leq m$. Hamiltonian paths in such graph have been studied by Itai, Papadimitiou and Szwarcfiter [ItaiPapadimitriouSzwarcfiter, 1982], and their results imply that between any two vertices there is a spanning path that repeats at most 2 elements. This shows that the Cayley graph of $\mathbb{Z}^2$ with standard generators, as well as the graph $\mathbb{Z} \times \{1, \ldots, n\}$, $n \geq 2$, have the quasi-Hamiltonian property. The rest of the proof consists of showing that any Cayley graph of an abelian group G, other than $\mathrm{Cay}(\mathbb{Z}, \{\pm 1\})$, contains one of the above graphs as a spanning subgraph. In other words, that for any Cayley graph $\mathrm{Cay}(G, S)$ as above, there exists a bijective 1-Lipschitz embedding whose domain is one of the graphs $\mathbb{Z}^2$ or $\mathbb{Z} \times \{1, \ldots, n\}$, for some $n \geq 2$. This will follow from an inductive argument, based on the proofs of Vászonyi [Vázsonyi, 1939] and Nash-Williams [Nash-Williams, 1959] of the existence of Hamiltonian double rays in the Cayley graph of any abelian group.

In order to formulate our results, we introduce some notation. Given $m \geq 1$, we denote by I_m the interval graph on m vertices. That is, the graph whose vertex set is $\{1, \ldots, m\}$ and where edges connect i with $i + 1$, for $0 \leq i < m$. More generally, for any integers $m_1, \ldots, m_s \geq 1$, we use the notation

$$\mathrm{Cube}(m_1, \ldots, m_s) \coloneqq I_{m_1} \times \cdots \times I_{m_s}$$

for the product graph of all the I_{m_i}'s (see Definition 2.2.4). When $s = 2$, we call $\mathrm{Cube}(m_1, m_2)$ a *grid graph*.

The existence of Hamiltonian paths between two vertices of a grid graph $\mathrm{Cube}(m_1, m_2)$ is studied in [ItaiPapadimitriouSzwarcfiter, 1982]. In particular, it is shown that obstructions to Hamiltonian-connectedness arise either from a parity issue, or by some particular configurations when either m_1 or m_2 are at most 3. Examples of such non-Hamiltonian-connected grid graphs are illustrated in Figure 2.3.

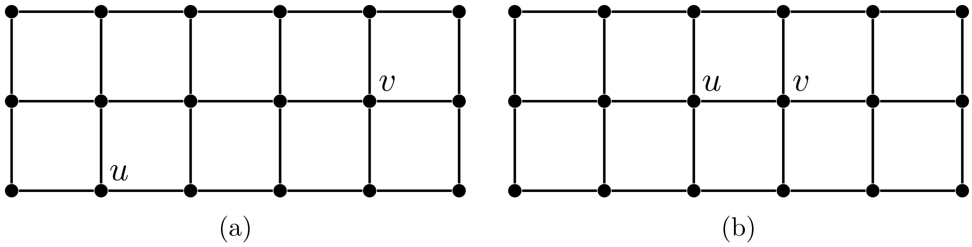


Figure 2.3 – There are no Hamiltonian paths from u to v in these grid graphs.

Lemma 2.4.10 (Hamiltonian-connectivity of grid graphs)**.** *Consider $m_1, m_2 \geq 2$. Then between any two vertices $s, t \in \mathrm{Cube}(m_1, m_2)$, there exists a spanning path of $\mathrm{Cube}(m_1, m_2)$ of length at most $|\mathrm{Cube}(m_1, m_2)| + 2$.*

Proof. A necessary and sufficient condition for the existence of a Hamiltonian path of $\mathrm{Cube}(m_1, m_2)$ between two vertices is provided in [ItaiPapadimitriouSzwarcfiter, 1982, Theorem 3.2], which depends only on whether the grid graph is bipartite, and on some particular cases where $m_1 \leq 3$ or $m_2 \leq 3$. This result implies that for any two pair of vertices $s, t \in \mathrm{Cube}(m_1, m_2)$, there is either a Hamiltonian path from s to t, or a Hamiltonian path from s to a vertex at distance at most 2 from t. ☐

Now we prove the higher-dimensional version of Lemma 2.4.10.

Lemma 2.4.11. *For any $r, m_1, \ldots, m_s \geq 1$, consider the graph*

$$\Gamma = \mathbb{Z}^r \times \mathrm{Cube}(m_1, \ldots, m_s),$$

where $\mathbb{Z}^r$ is identified with its Cayley graph with respect to canonical generators.

Suppose that Γ is not a line, that is, either $r \geq 2$, or $r = 1$ and $s \geq 1$. Then there exists a constant $M \geq 0$ such that for any $n_1, \ldots, n_r \geq 1$, the induced subgraph

$$R = \mathrm{Cube}(n_1, \ldots, n_r) \times \mathrm{Cube}(m_1, \ldots, m_s) \subseteq \mathbb{Z}^r \times \mathrm{Cube}(m_1, \ldots, m_s)$$

has a spanning path of length at most $|R| + M$ between any two vertices $s, t \in R$.

Proof. We start by noting that if $r = s = 1$ then $\Gamma \cong \mathbb{Z} \times I_{m_1}$, and that if $r = 2$ and $s = 0$ then $\Gamma \cong \mathbb{Z}^2$. In both cases R is isomorphic to a grid graph of appropriate dimensions, and hence the result follows from Lemma 2.4.10 with $M = 2$. These are the base cases for an inductive argument, which we explain now.

Let us first consider the case $r = 1$ and suppose that $s \geq 2$, so that we have

$$\Gamma \cong \mathbb{Z} \times \mathrm{Cube}(m_1, \ldots, m_s),$$

For $\mathrm{Cube}(m_{s-1}, m_s)$ consider the Hamiltonian path P that starts at $(1,1)$, traverses the edges of $I_{m_{s-1}} \times \{1\}$ until it reaches $(m_{s-1}, 1)$, then crosses to $(m_{s-1}, 2)$ and continues in a similar way traversing one copy of $I_{m_{s-1}}$ at the time, until it finally reaches either (m_{s-1}, m_s) or $(m_{s-1}, 1)$, after having visited all vertices of the graph $I_{m_{s-1}} \times I_{m_s}$.

Define the function

$$h : \mathrm{Cube}(m_{s-1}, m_s) \to \{1, \ldots, m_{s-1} \cdot m_s\}$$

which assigns to each pair $(j,k) \in \mathrm{Cube}(m_{s-1}, m_s)$ its unique position

$$h(j,k) \in \{1, \ldots, m_{s-1} \cdot m_s\}$$

in the path P. That is, if we write $P = P_0, P_1, \ldots, P_\ell$ as a sequence of vertices, then

$$(j,k) = P_{h(j,k)}, \quad \text{for } (j,k) \in \mathrm{Cube}(m_{s-1}, m_s).$$

Note that as P is a Hamiltonian path, the function h is well defined and bijective.

With the above, we can see that any subgraph

$$R = I_n \times \mathrm{Cube}(m_1, \ldots, m_s)$$

has a spanning subgraph isomorphic to another one of same shape in the graph

$$\mathbb{Z} \times \mathrm{Cube}(m_1, \ldots, m_{s-2}, m_{s-1}m_s).$$

Indeed, it suffices to use the function h to map any element $(i, j_1, j_2, \ldots, j_s) \in R$ into

$$(i, j_1, \ldots, j_{s-2}, h(j_{s-1}, j_s)) \in \mathbb{Z} \times \mathrm{Cube}(m_1, \ldots, m_{s-2}, m_{s-1}m_s).$$

This construction is illustrated in Figure 2.4 for $s = 2$.

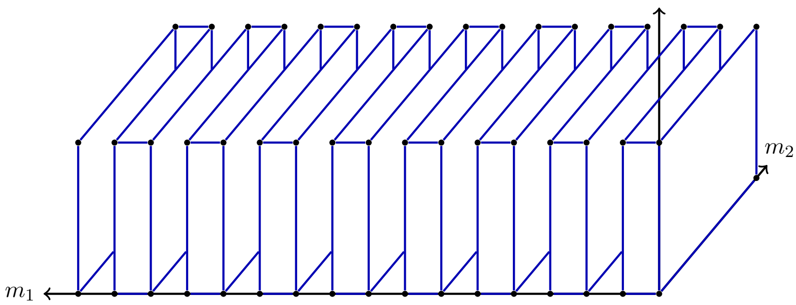


Figure 2.4 – Inductive step of the proof of Lemma 2.4.11.

Then, thanks to the induction hypothesis, this new subgraph has spanning paths of length at most $|R| + M$ between any pair of vertices, and so the same holds for R. This concludes the induction for the case $r = 1$.

For the remaining cases $r \geq 2$, a very similar induction proves the result. Indeed, now we repeat the argument using a Hamiltonian path between opposite corners of a grid subgraph, in order to find a spanning subgraph which is (isomorphic to) a similar subgraph on a graph with a lower number of finite factors s, or lower free rank r. Together with the already proved base case of $r = 2$ and $s = 0$, this finishes the inductive argument. □

The above implies that for any $r \geq 1$ and $m_1, \ldots, m_s \geq 1$ as in the hypotheses of Lemma 2.4.11, the Cayley graph of

$$\mathbb{Z}^r \times \mathbb{Z}/m_1\mathbb{Z} \times \cdots \mathbb{Z}/m_s\mathbb{Z}$$

with standard generators has the quasi-Hamiltonian property. In order to generalize this to hold for *every* Cayley graph, we will use the following result about the structure of generating sets of abelian groups.

Lemma 2.4.12 ([Nash-Williams, 1959, Lemma 3])**.** *Let B be an infinite abelian group with free abelian rank* $\operatorname{rank}(B) = r \geq 1$*, and S any finite generating set. Then we can label the elements of S as $a_1, \ldots, a_r, b_1, \ldots, b_s$ and find positive integers $m_1, \ldots, m_s$ such that each element $g \in B$ is uniquely expressible as*

$$g = \sum_{i=1}^{r} p_i a_i + \sum_{j=1}^{s} q_j b_j,$$

where the $p_i, q_j \in \mathbb{Z}$ and for each $j = 1, \ldots, s$, we have $0 \leq q_j < m_j$.

Lemma 2.4.13. *Under the assumptions of Lemma 2.4.12, the function*

$$\begin{aligned} \varphi : \mathbb{Z}^r \times \operatorname{Cube}(m_1, \ldots, m_s) &\to B \\ (p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s) &\mapsto \sum_{i=1}^{r} p_i a_i + \sum_{j=1}^{s} q_j b_j \end{aligned}$$

is a bijective 1-Lipschitz embedding of $\mathbb{Z}^r \times \operatorname{Cube}(m_1, \ldots, m_s)$ onto the Cayley graph $\operatorname{Cay}(B, S)$*. Moreover, r can be chosen to be any integer between* $\min\{2, \operatorname{rank}(B)\}$ *and* $\operatorname{rank}(B)$*.*

Proof. The first sentence follows from Lemma 2.4.12, while the second one follows from an analogous inductive argument to the one used in the proof of Lemma 2.4.11. □

Proposition 2.4.14. *With the exception of* $\operatorname{Cay}(\mathbb{Z}, \{\pm 1\})$*, any Cayley graph of an infinite finitely generated abelian group is quasi-Hamiltonian.*

Proof. Let (B, S) be an infinite finitely generated abelian group, different from $(\mathbb{Z}, \{\pm 1\})$. Then Lemma 2.4.13 tells us that for some $r, m_1, \ldots, m_s \geq 1$, there is a bijective 1-Lipschitz embedding of $\mathbb{Z}^r \times \operatorname{Cube}(m_1, \ldots, m_s)$ onto $\operatorname{Cay}(B, S)$. In other words, tha former graph is a spanning subgraph of the latter one.

Now using Lemma 2.4.11, there exists a constant $M \geq 0$ such that any rectangle

$$R_n = [-n, n]^r \times \mathrm{Cube}(m_1, \ldots, m_s),$$

for $n \geq 1$, has spanning paths of length at most $|R_n| + M$ between any pair of vertices. Any ball of $\mathrm{Cay}(B, S)$ centered at the origin is contained in R_n for n sufficiently large, and the above implies that we can find paths from the origin to any other vertex with paths of length at most $|R_n| + M$. This shows that $\mathrm{Cay}(B, S)$ has the quasi-Hamiltonian property. □

Proposition 2.4.15. *For (A, S_A) a group of unbounded depth, and (B, S_B) any finitely generated abelian group, the lamplighter group $A \wr B$ has unbounded depth with respect to S_{std}.*

Proof. The case where $(B, S_B) = (\mathbb{Z}, \pm 1)$ follows from the original example of Cleary and Taback [ClearyTaback, 2005a], or alternatively it can be seen as a particular case of Proposition 2.3.3 for lamplighters over trees.

In any other case, Corollary 2.4.14 shows that (B, S_B) has the quasi-Hamiltonian property and hence Lemma 2.4.5 implies that the group $(A \wr B, S_{\mathrm{std}})$ has unbounded depth. □

A natural question is whether the claim of Lemma 2.4.13 holds for some non-abelian groups. That is, whether there is a bijective 1-Lipschitz embedding of a graph of the form $\mathbb{Z}^r \times \mathrm{Cube}(m_1, \ldots, m_s)$ onto all of its Cayley graphs. As shown by Nash-Williams, any such graph must admit Hamiltonian double ray. With respect to this property, Thomassen has shown that it holds for any Cayley graph of the form $\mathrm{Cay}(G, S \cup S^2)$, where G is a 1-ended group and S is a finite generating set [Thomassen, 1978]. In [Seward, 2014], Seward studies *translation-like actions* of free groups. In particular, he characterizes groups with finitely many ends as those which admit a transitive translation-like action of $\mathbb{Z}$, and shows that this is equivalent to having a Cayley graph that admits a Hamiltonian double ray. In the same paper, it is mentioned in Problem 4.8 that the existence of a Hamiltonian double ray in *every* Cayley graph of an infinite finitely generated group with finitely many ends is an open question. With respect to our question of extending Lemma 2.4.13, a first approach could be to study transitive translation-like actions of $\mathbb{Z}^r$, $r \geq 2$.

2.4.5 Remarks

Unlike the original example of Cleary and Taback, the dead ends of $A \wr B$ we found using quasi-Hamiltonian Cayley graphs of the base group B are of bounded retreat depth (Definition 2.2.8). Hence the following question remains open.

Question 2.4.16. Suppose (A, S_A) has unbounded retreat depth. Does every lamplighter group $A \wr B$ admit a standard generating set with unbounded retreat depth?

So far we have seen that this holds for B a free group with a free generating set (Proposition 2.3.3), but to our knowledge there are no other known examples. An answer to this question would be of interest even in the case of particular base groups, as for example $B = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $B = \mathbb{Z}^2$.

Another observation is that in the groups we have studied so far, the constant M from the definition of the quasi-Hamiltonian property is at most 2. We have not been able to find examples of a Cayley graph for which this constant is necessarily bigger. More generally, we ask the following.

Question 2.4.17. Given an arbitrary $n \geq 1$, does there exist a group B together with a generating set S_B such that $\mathrm{Cay}(B, S_B)$ satisfies Definition 2.4.4 with $M = n$ but not with $M = n - 1$?

2.5 Lamplighters over free products of finite groups

Our results in the previous sections concern lamplighter groups with unbounded depth with respect to standard generating sets. We begin this section by showing that this is not always the case, with an example of a lamplighter group over the free product $\mathbb{Z}/8\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ that has uniformly bounded depth with respect to standard generators.

Next, we characterize which standard generators of lamplighter groups over free products of finite groups have this property. For this, we define for a finite group G with a generating set S_G its *Hamiltonian difference* $\mathscr{H}(G, S_G)$ (Definition 2.5.2), which measures how much shorter minimal spanning cycles are than minimal spanning paths inside $\mathrm{Cay}(G, S_G)$. The main result of this section says that a lamplighter group over the free product $H * K$, where (H, S_H) and (K, S_K) are finite groups with their respective generating sets, has uniformly bounded depth if and only if $\mathscr{H}(H, S_H) + \mathscr{H}(K, S_K) \geq 1$ (Theorem 2.5.3).

In what follows we use the following observation. For H, K finite groups, consider their free product $H * K$ with a generating set of the form $S_H \cup S_K$, where S_H and S_K are generating sets of H and K, respectively. We can partition $\mathrm{Cay}(H * K, S_H \cup S_K)$ as follows. Any element $x \in H * K$ belongs to a copy of $\mathrm{Cay}(H, S_H)$ which, when removed, divides the Cayley graph $\mathrm{Cay}(H * K, S_H \cup S_K)$ into $|H|$ connected components that we number $P_0, \dots, P_{|H|-1}$. To each of these sets P_i we add the unique vertex of the original copy of $\mathrm{Cay}(H, S_K)$ to which it is connected in $\mathrm{Cay}(H * K, S_H \cup S_K)$, and we call them the *petals* associated with this copy. A similar decomposition holds when considering x as forming part of a copy of $\mathrm{Cay}(K, S_K)$, now obtaining $|K|$ petals associated with each copy of this finite subgraph. This decomposition by petals will allow us to compare the solutions to different instances of the TSP in $\mathrm{Cay}(H * K, S_H \cup S_K)$, and hence the word metric of the associated lamplighter group.

2.5.1 A lamplighter group with uniformly bounded depth for standard generating sets

Fix the lamps group $\mathbb{Z}/2\mathbb{Z} = \langle a \mid a^2 \rangle$, and consider the groups $H = \mathbb{Z}/8\mathbb{Z} = \langle b \mid b^8 \rangle$ and $K = \mathbb{Z}/2\mathbb{Z} = \langle c \mid c^2 \rangle$. We will study the wreath product $\mathbb{Z}/2\mathbb{Z} \wr (H * K)$ with standard generating set $S_{\mathrm{std}} = \{a, b, c\}^{\pm 1}$. On what follows we prove that the dead end depth of $\mathbb{Z}/2\mathbb{Z} \wr (H * K)$ is uniformly bounded with respect to S_{std}.

The Cayley graph of $\mathbb{Z}/8\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ with respect to the generating set $\{b, c\}$ is formed by octagons (the Cayley graph of $\mathrm{Cay}(\mathbb{Z}/8\mathbb{Z}, b)$) joined together by the generator c. This graph contains no odd cycles and hence is bipartite so that for any edge, one of its extremes is strictly closer to the identity than the other one. In particular, if we order cyclically the vertices of an octagon as $v_0, \ldots, v_7$ with v_0 being the closest to the identity element, then v_4 is the furthermost one.

As explained at the beginning of this section, once we have a free product we can consider the petal partition induced by each vertex. In this case, each octagon in $\mathrm{Cay}(\mathbb{Z}/8\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}, \{b, c\})$ defines a partition of the Cayley graph into 8 subsets, which we call petals and denote by $P_0, P_1, \ldots, P_7$, chosen in a cyclic order so that P_0 is the component containing the identity and P_4 is the furthermost one. This partition is illustrated in Figure 2.5.

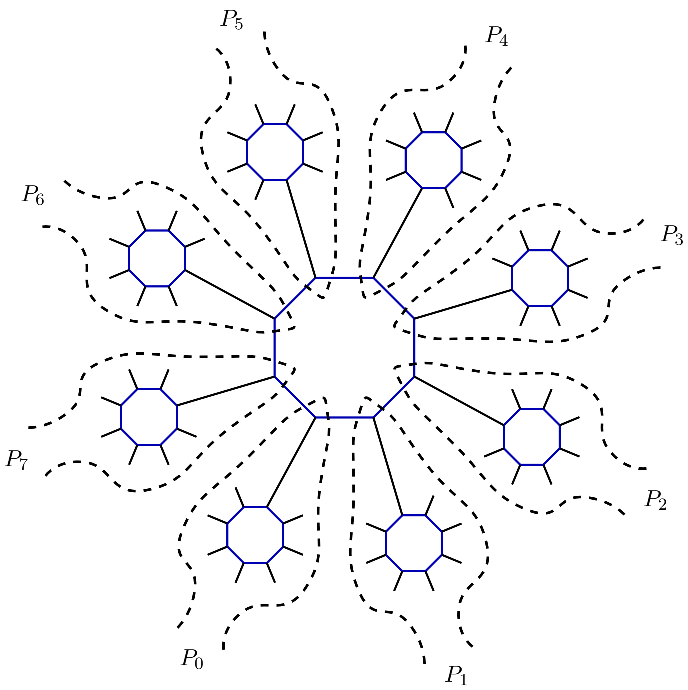


Figure 2.5 – Each octagon defines a "partition by petals" of $\mathrm{Cay}(\mathbb{Z}/8\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}, \{b, c\})$.

Now we are ready to prove that the depth of any element of $\mathbb{Z}/2\mathbb{Z} \wr (H * K)$ is uniformly bounded. Indeed, consider an arbitrary element $g \in \mathbb{Z}/2\mathbb{Z} \wr (H * K)$, and write $g = (f, p)$ where

$$f \in \bigoplus_{\mathbb{Z}/8\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \text{ and } p \in \mathbb{Z}/8\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.$$

As we are trying to find a bound on the depth of g, we lose no generality if we suppose that $\|g\|_{S_{\mathrm{std}}} > 10$; as $\mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z}/8\mathbb{Z} * \mathbb{Z}/2\mathbb{Z})$ is an infinite group, the depth of any element is finite and hence the depth of elements on $B_{S_{\mathrm{std}}}(e, 10)$ is uniformly bounded.

According to our notation, p corresponds to the position of the lamplighter in $\mathrm{Cay}(B, S_B)$ for the element g, where $B = \mathbb{Z}/8\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and $S_B = \{b, c\}$. We divide the rest of the proof on two cases.

Case 1: Suppose there exists $p' \in B_{S_B}(p, 10)$ with $f(p') = e_{\mathbb{Z}/2\mathbb{Z}}$. In other words, there is an unlit lamp at distance at most 10 from p. In such a case we multiply by a word of length at most 21 which moves the lamplighter position from p to p', lights the corresponding lamp at p', and returns to p. This will forcefully result in an element of word length at least 1 more than $\|g\|_{S_{\text{std}}}$. Hence, the depth of g is bounded above by 21.

Case 2: $f(p') = a$ for all $p' \in B_{S_B}(p, 10)$. Now we suppose that all lamps within a 10-neighborhood of p are lit. We look at the octagon to which p belongs, and number its vertices $v_0, v_1, \ldots, v_7$ according to order induced by the cyclic generator b. Similarly, we consider the partition by petals $P_0, P_1, \ldots, P_7$ determined by this octagon. In the case we are considering, there are lamps lit at each petal and so

$$\|g\|_{S_{\text{std}}} = |\text{supp}(f)| + \ell_0 + \ell_1 + \cdots + \ell_7 + 7 + \min\{d_B(p, v_1), d_B(p, v_7)\},$$

where

$$\ell_0 = \text{TS}(e_B, v_0, f|_{P_0}), \ \ell_i = \text{TS}(v_i, v_i, f|_{P_i}) \ \text{for } i = 1, \ldots, 7,$$

are the lengths of minimal paths traversing the support of f at each petal and finishing in the respective vertices v_i. This is is illustrated in Figure 2.6. Note that we have $\min\{d_B(p, v_1), d_B(p, v_7)\} \leq 3$ so that

$$\|g\|_{S_{\text{std}}} \leq |\text{supp}(f)| + \ell_0 + \ell_1 + \cdots + \ell_7 + 10.$$

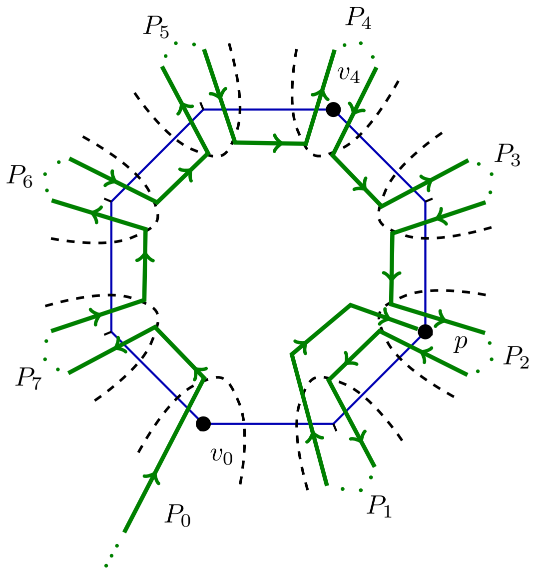


Figure 2.6 – An optimal path for g must visit all petals and return to p.

Now we will explain how to increase the word length of g by multiplying by a bounded number of generators. Look in more detail at the octagon p is together with its petal P_4. This is defined by a new octagon whose vertices we name $v'_0, v'_1, \ldots, v'_7$ and petals $P'_0, P'_1, \ldots, P'_7$ similarly as we did before. By using at most 9 generators of B, we can move the position of the lamplighter from p to the vertex v_4 of the octagon defining P_4. Indeed, we need to traverse at most 4 edges of the octagon of p, then the generator c to pass to the next octagon, and then 4 more edges. We call this new element g'. This is depicted in Figure 2.7.

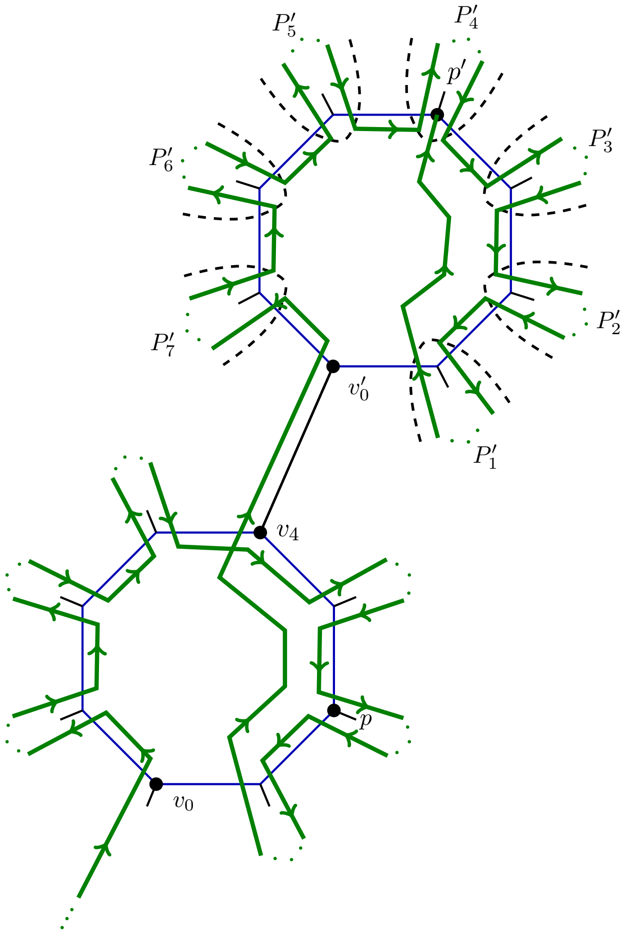


Figure 2.7 – The new position p' of the lamplighter for the element g', with $d_{S_{\mathrm{std}}}(g, g') \leq 9$.

Note that defining analogously the lengths of minimal paths passing through the petals P'_i of the new octagon,

$$\ell'_i = \mathrm{TS}\left(v'_i, v'_i, f|_{P'_i}\right), \; i = 1, 2, \ldots, 7,$$

we have $\ell_4 = \ell'_1 + \ell'_2 + \cdots + \ell'_7 + 10$. Indeed, the path of length ℓ_4 covering the elements of P_4 must cross to v'_0, pass through each petal P'_i, $i = 1, \ldots, 7$, while traversing the octagon to finally return to v'_0 and afterwards to v_4. On the other hand, we see that

$$\begin{aligned}\|g'\|_{S_{\mathrm{std}}} =& |\mathrm{supp}(f)| + \ell_0 + \ell_1 + \ell_2 + \ell_3 + \ell_5 + \ell_6 + \ell_7 + 10 + 1+ \\ &+ \ell'_1 + \ell'_2 + \ell'_3 + \ell'_4 + \ell'_5 + \ell'_6 + \ell'_7 + 10 \\ =& |\mathrm{supp}(f)| + \ell_0 + \ell_1 + \cdots + \ell_7 + 11 \\ \geq& \|g\|_{S_{\mathrm{std}}} + 1.\end{aligned}$$

This proves that g has depth at most 9, and hence finishes the proof.

The existence of generating sets for lamplighter groups with uniformly bounded depth has been proved by Warshall [Warshall, 2008]. However such generating sets are not standard ones, and hence the example explained in this Subsection (and more generally Corollary 2.5.7 below) seem to provide the first example of lamplighter groups with uniformly bounded depth with respect to standard generators.

Corollary 2.5.1. *Let (A, S_A) be a lamps group of unbounded depth. Then there exist finite groups (H, S_H), (K, S_K) for which $(A \wr (H * K), S_{\text{std}})$ has uniformly bounded depth.*

2.5.2 A general characterization

Recall that given any two vertices v, w of a finite connected graph Γ, we denote by $\operatorname{TS}(v, w, \Gamma)$ the minimal length of a path in Γ which starts at v, ends at w and visits all vertices of Γ.

Definition 2.5.2. Let G be a finite group and S_G a finite set. We define

$$\mathscr{H}(G, S_G) = \max_{g \in G \setminus \{e_G\}} \left\{ \operatorname{TS}(e_G, g, G) \right\} - \operatorname{TS}(e_G, e_G, G),$$

where as usual the TSP is considered in the Cayley graph $\operatorname{Cay}(G, S_G)$.

Note that for any generator $g \in S_G$, we have

$$\operatorname{TS}(e_G, e_G, G) \leq \operatorname{TS}(e_G, g, G) + 1,$$

so that it always holds that

$$\mathscr{H}(G, S_G) \geq -1.$$

Moreover, this lower bound is attained when $\operatorname{Cay}(G, S_G)$ is Hamiltonian-connected. We now state the main result of this section, which characterizes standard generating sets of unbounded depth for lamplighter over a free product of finite groups, in terms of their Hamiltonian differences.

Theorem 2.5.3. *Let (H, S_H), (K, S_K) be finite groups together with finite symmetric generating sets. Consider the free product $H * K$ with generating set $S_H \cup S_K$, and the lamplighter group $A \wr (H * K)$ with standard generating set S_{std}, where (A, S_A) is the lamps group of unbounded depth. Then $A \wr (H * K)$ has uniformly bounded depth with respect to S_{std} if and only if*

$$\mathscr{H}(H, S_H) + \mathscr{H}(K, S_K) \geq 1. \tag{2.3}$$

Proof. Let us suppose first that Equation (2.3) holds, and let us prove that $A \wr (H * K)$ has uniformly bounded depth. Begin by choosing elements $v \in H \setminus \{e_H\}$, $w \in K \setminus \{e_K\}$ with

$$\operatorname{TS}(e_H, v, H) = L_H := \max_{v' \in H \setminus \{e_H\}} \{\operatorname{TS}(e_H, v', H)\},$$

and

$$\operatorname{TS}(e_K, w, K) = L_K := \max_{w' \in K \setminus \{e_K\}} \{\operatorname{TS}(e_K, w', K)\}.$$

Our hypothesis is that

$$L_H + L_K \geq \mathrm{TS}\,(e_H, e_H, H) + \mathrm{TS}\,(e_K, e_K, K) + 1. \tag{2.4}$$

Note that Equation (2.4) implies that either

$$L_H \geq \mathrm{TS}\,(e_H, e_H, H) + 1,$$

or

$$L_K \geq \mathrm{TS}\,(e_K, e_K, K) + 1.$$

Without loss of generality, we suppose this is the case for L_H.

To simplify notation, denote $G \coloneqq H * K$ and $S_G = S_H \cup S_K$. Let $g = (f, p) \in A \wr G$ be any element. If there exists $p' \in B_{S_G}(p, 2|K| + 2|H|)$ with $f(p') = e_A$, then we can increase the word length of g by moving the lamplighter to p', changing the state of this lamp to a non-trivial element, and then return to p. This uses at most $4(|K| + |H|) + 1$ generators and since H and K are finite this is a constant.

Now assume that for all $p' \in B_{S_G}(p, 2|K| + 2|H|)$ we have $f(p') \neq e_A$, so that any path in $\mathrm{Cay}(G, S_G)$ evaluating to g must visit all the elements in said ball at least once.

Consider the copy of $\mathrm{Cay}(H, S_H)$ to which p belongs, and the associated petals $P_0, \dots, P_{|H|-1}$, where P_0 is the petal containing the identity e_G. In the same way, we number the vertices of this copy of H as $v_0, \dots, v_{|H|-1}$ according to their associated petal. Then the word length of g can be expressed as

$$\|g\|_{S_{\mathrm{std}}} = \sum_{z \in \mathrm{supp}(f)} \|f(z)\|_{S_A} + \ell_0 + \ell_1 + \cdots + \ell_{|H|-1} + \mathrm{TS}\,(e_H, v(p), H)\,,$$

where $\ell_0 = \mathrm{TS}\,(e_H, v_0, f|_{P_0})$, $\ell_i = \mathrm{TS}\,(v_i, v_i, f|_{P_i})$, $i = 1, \dots, |H| - 1$, and $v(p)$ is the vertex of $\mathrm{Cay}(H, S_H)$ that coincides with the position of p inside this copy of H. In particular, we have

$$\|g\|_{S_{\mathrm{std}}} \leq \sum_{z \in \mathrm{supp}(f)} \|f(z)\|_{S_A} + \ell_0 + \ell_1 + \cdots + \ell_{|H|-1} + L_H.$$

Now consider the element $g' = gv(p)^{-1}vwv$. That is, we are moving the position of the lamplighter from p to the vertex v (which maximizes the value of $\mathrm{TS}\,(e_H, v, H)$), then to the vertex w (which maximizes the value of $\mathrm{TS}\,(e_K, w, K)$) of the corresponding copy of $\mathrm{Cay}(K, S_K)$, and finally again to the vertex v of the new copy of $\mathrm{Cay}(H, S_H)$.

As before, the vertices of $\mathrm{Cay}(K, S_K)$ define a partition by petals $P'_0, P'_1, \dots, P'_{|K|-1}$ of $\mathrm{Cay}(K, S_K)$, where the identity element e_K belongs to P'_0. Again, we number the vertices of this copy of $\mathrm{Cay}(K, S_K)$ by $w'_0, \dots, w'_{|K|-1}$ according to the petal they define. Defining

$$\ell'_j = \mathrm{TS}\left(w'_j, w'_j, f|_{P'_j}\right),\ j = 1, \dots, |K| - 1,$$

it follows that if $v = v_i$, for some $i \in \{1, \ldots, |H|-1\}$, then

$$\ell_i = \mathrm{TS}\,(e_K, e_K, K) + \ell'_1 + \cdots + \ell'_{|K|-1}.$$

Similarly, the new copy of $\mathrm{Cay}(H, S_H)$ gives a new partition of $\mathrm{Cay}(K, S_K)$ into petals $P''_0, P''_1, \ldots, P''_{|H|-1}$ with $e_K \in P''_0$ and the vertices of H numbered as v''_m according to the petal they belong to. Say that $w = w'_r$ for some $r \in \{1, \ldots, |K|-1\}$. Then

$$\ell'_r = \mathrm{TS}\,(e_H, e_H, H) + \ell''_1 + \cdots + \ell''_{|H|-1},$$

where we defined $\ell''_h = \mathrm{TS}\left(v''_h, v''_h, f|_{P''_h}\right)$, $h = 1, \ldots, |H|-1$.

We can express the word length of g' in terms of all these values. Indeed, we have

$$\begin{aligned}\|g'\|_{S_{\mathrm{std}}} = &\sum_{z\in\mathrm{supp}(f)} \|f(z)\|_{S_A} + \sum_{j\neq i} \ell_j + \mathrm{TS}\,(e_H, v, H)\ + \\ &+ \mathrm{TS}\,(e_K, w, K) + \sum_{h\neq r} \ell'_h + \mathrm{TS}\,(e_H, v, H) + \ell''_1 + \cdots + \ell''_{|H|-1}.\end{aligned}$$

Combining the last three equations together with Equation (2.4), we obtain

$$\begin{aligned}\|g'\|_{S_{\mathrm{std}}} =& \sum_{z\in\mathrm{supp}(f)} \|f(z)\|_{S_A} + \sum_{j\neq i} \ell_j + \mathrm{TS}\,(e_H, v, H)\ + \\ &+ \mathrm{TS}\,(e_K, w, K) + \sum_{h\neq r} \ell'_h + \mathrm{TS}\,(e_H, v, H) + \ell''_1 + \cdots + \ell''_{|H|-1} \\ =& \sum_{z\in\mathrm{supp}(f)} \|f(z)\|_{S_A} + \sum_{j\neq i} \ell_j + L_H + L_K + \sum_{h\neq r} \ell'_h + L_H + \ell''_1 + \cdots + \ell''_{|H|-1} \\ \geq& \sum_{z\in\mathrm{supp}(f)} \|f(z)\|_{S_A} + \sum_{j\neq i} \ell_j + \mathrm{TS}\,(e_H, e_H, H) + \mathrm{TS}\,(e_K, e_K, K) + 1 + \\ &+ \sum_{h\neq r} \ell'_h + L_H + \ell''_1 + \cdots + \ell''_{|H|-1} \\ =& \sum_{z\in\mathrm{supp}(f)} \|f(z)\|_{S_A} + \sum_{j\neq i} \ell_j + \mathrm{TS}\,(e_K, e_K, K) + 1 + \sum_{h\neq r} \ell'_h + L_H + \ell'_r \\ =& \sum_{z\in\mathrm{supp}(f)} \|f(z)\|_{S_A} + \sum_{j\neq i} \ell_j + \mathrm{TS}\,(e_K, e_K, K) + 1 + \sum_{h} \ell'_h + L_H \\ =& \sum_{z\in\mathrm{supp}(f)} \|f(z)\|_{S_A} + \sum_{j\neq i} \ell_j + \ell_i + 1 + L_H \\ =& \sum_{z\in\mathrm{supp}(f)} \|f(z)\|_{S_A} + \sum_{j} \ell_j + L_H + 1 \\ \geq& \|g\|_{S_{\mathrm{std}}} + 1.\end{aligned}$$

In the process of obtaining g' we multiplied by at most $2|H| + |K|$ generators, and so we conclude that the depth of any element is bounded by a uniform constant. This proves the first direction of the proposition.

Now let us prove that if Equation (2.3) does not hold then the associated lamplighter group has unbounded depth. From the hypothesis we deduce the inequalities

$$\mathrm{TS}\,(e_H, e_H, H) - 1 \leq \max_{v \in H \setminus \{e_H\}} \{\mathrm{TS}\,(e_H, v, H)\} \leq \mathrm{TS}\,(e_H, e_H, H) + 1,$$

and

$$\mathrm{TS}\,(e_K, e_K, K) - 1 \leq \max_{w \in K \setminus \{e_K\}} \{\mathrm{TS}\,(e_K, w, K)\} \leq \mathrm{TS}\,(e_K, e_K, K) + 1.$$

With this, the possible values for these two maximums are illustrated in Table 2.1. Here we denote $L_H = \max_{v \in H \setminus \{e_H\}} \{\mathrm{TS}\,(e_H, v, H)\}$ and

$L_K = \max_{w \in K \setminus \{e_K\}} \{\mathrm{TS}\,(e_K, w, K)\}$. Each possible combination of values of L_H and L_K

L_K \ L_H	$\mathrm{TS}\,(e_H, e_H, H) - 1$	$\mathrm{TS}\,(e_H, e_H, H)$	$\mathrm{TS}\,(e_H, e_H, H) + 1$
$\mathrm{TS}\,(e_K, e_K, K) - 1$	Case 2	Case 2	Case 1
$\mathrm{TS}\,(e_K, e_K, K)$	Case 2	Case 2	Impossible
$\mathrm{TS}\,(e_K, e_K, K) + 1$	Case 1	Impossible	Impossible

Table 2.1 – Possible values for the solutions of the TSP inside each finite graph.

will be covered in two separate cases, as shown in the table. Recall that for convenience, we defined $G = H * K$ and $S_G = S_H \cup S_K$.

Case 1. Suppose that $L_H = \mathrm{TS}\,(e_H, e_H, H) + 1$, so that we must have

$$L_K = \mathrm{TS}\,(e_K, e_K, K) - 1.$$

Choose $v \in H$ such that $\mathrm{TS}\,(e_H, v, H) = L_H$.

Consider $a \in (A, S_A)$ of depth at least n, and as usual define the element $g = (f, v)$, where $f(x) = a$ if $\|x\|_K \leq n$ and $f(x) = e_A$ otherwise. We will prove that g has depth at least $n - 1$. Similar to how we approached the case of free groups, changing the lamp states cannot increase word length so the proof will follow from the following claim.

We claim that for any $x \in G$ with $1 \leq \|x\|_{S_G} \leq n - 1$, we have $\|gx\|_{S_{\mathrm{std}}} \leq \|g\|_{S_{\mathrm{std}}} - 1$ if x finishes with an element of $K \setminus \{e_K\}$ and $\|gx\|_{S_{\mathrm{std}}} \leq \|g\|_{S_{\mathrm{std}}}$ if x finishes with an element of $H \setminus \{e_H\}$.

Indeed, let us do an inductive proof. If $x \in H$, then

$$\|gx\|_{S_{\mathrm{std}}} = \|g\|_{S_{\mathrm{std}}} + \mathrm{TS}\,(e_H, x, H) - \mathrm{TS}\,(e_H, v, H) \leq \|g\|_{S_{\mathrm{std}}}.$$

Similarly, if $x \in K$ then

$$\begin{aligned} \|gx\|_{S_{\mathrm{std}}} &= \|g\|_{S_{\mathrm{std}}} + \mathrm{TS}\,(e_K, x, K) - \mathrm{TS}\,(e_K, e_K, K) \\ &\leq \|g\|_{S_{\mathrm{std}}} + \mathrm{TS}\,(e_H, e_H, H) - L_H \\ &= \|g\|_{S_{\mathrm{std}}} - 1 \\ &\leq \|g\|_{S_{\mathrm{std}}}. \end{aligned}$$

Now suppose that x is of the form $x'hk$, for $h \in H \backslash \{e_H\}$, $k \in \backslash \{e_K\}$. Then looking at the petal decomposition of $\mathrm{Cay}(H, S_H)$ and using the inductive hypothesis,

$$\begin{aligned} \|gx\|_{S_{\mathrm{std}}} &= \|gx'hk\|_{S_{\mathrm{std}}} \\ &= \|gx'h\|_{S_{\mathrm{std}}} + \mathrm{TS}\,(e_K, k, K) - \mathrm{TS}\,(e_K, e_K, K) \\ &\leq \|g\|_{S_{\mathrm{std}}} - 1. \end{aligned}$$

On the other hand, if x is of the form $x'kh$ for $h \in H \backslash \{e_H\}$, $k \in \backslash \{e_K\}$. Similarly to the above we have,

$$\begin{aligned} \|gx\|_{S_{\mathrm{std}}} &= \|gx'kh\|_{S_{\mathrm{std}}} \\ &= \|gx'k\|_{S_{\mathrm{std}}} + \mathrm{TS}\,(e_H, h, H) - \mathrm{TS}\,(e_H, e_H, H) \\ &\leq \|gx'k\|_{S_{\mathrm{std}}} + 1 \\ &\leq \|g\|_{S_{\mathrm{std}}}. \end{aligned}$$

This finishes the proof of the first case. By symmetry of H and K, the proof for the case where $L_K = \mathrm{TS}\,(e_K, e_K, K) + 1$ is completely analogous.

Case 2. Now suppose that $L_H \leq \mathrm{TS}\,(e_H, e_H, H)$ and $L_K \leq \mathrm{TS}\,(e_K, e_K, K)$. A similar inductive argument to the one given in the first case proves that if we define g as in Case 1, now for any $x \in K$ with $1 \leq \|x\|_{S_K} \leq n$ we have $\|gx\|_{S_{\mathrm{std}}} \leq \|g\|_{S_{\mathrm{std}}}$.

With this, we see that the element g constructed has depth at least $n-1$. As n was arbitrary, we conclude that $A \wr (H * K)$ has unbounded depth with respect to S_{std}. □

If $\mathrm{Cay}(G, S_G)$ is a cycle, then the value of $\mathrm{TS}\,(e_G, e_G, G)$ is always equal to $|G|$, which is attained with a path starting at e_G and traversing the cycle. On the other hand, the value of $\max_{g \in G \backslash \{e_G\}} \left\{ \mathrm{TS}\,(e_G, g, G) \right\}$ is

$$|G| - 1 + \left\lfloor \frac{|G|}{2} \right\rfloor - 1 = |G| + \left\lfloor \frac{|G|}{2} \right\rfloor - 2.$$

This is the length of a path starting at e_G, doing the cycle up to the last vertex before returning to e_G, and then going back to an element $g \in G$ at distance $\left\lfloor \frac{|G|}{2} \right\rfloor$ from the identity.

The above implies that $\mathscr{H}(G, S_G) = \left\lfloor \frac{|G|}{2} \right\rfloor - 2$, and so in particular we have that $\mathscr{H}(\mathbb{Z}/8\mathbb{Z}, \{b\}) + \mathscr{H}(\mathbb{Z}/2\mathbb{Z}, \{b\}) = 1$, so that the example of Subsection 2.5.1 is consistent with Theorem 2.5.3.

More generally, for any pair of cyclic groups (H, S_H) and (K, S_K), with cyclic generating sets, Condition (2.3) holds if and only if

$$\left\lfloor \frac{|H|}{2} \right\rfloor + \left\lfloor \frac{|K|}{2} \right\rfloor \geq 5.$$

Denoting the orders of H and K by o_H and o_K, respectively, we have that Theorem 2.5.3 implies the lamplighter over $H * K$ has unbounded depth with respect to the standard generating set if and only if

1. $(o_H, o_K) \in \{(6,4),(6,5),(6,6),(7,4),(7,5),(7,6),(7,7)\}$, or
2. $o_H \geq 8$ and $o_K \geq 2$.

Example 2.5.4. Consider $H = \mathbb{Z}/4\mathbb{Z} = \langle b \rangle$ and $K = \mathbb{Z}/4\mathbb{Z} = \langle c \rangle$, so that according to Corollary 2.5.7 the lamplighter group $A \wr (H * K)$ has unbounded depth. By following the proof of Theorem 2.5.3, it is possible to see that for any element $x \in \langle b^2, c^2 \rangle \leqslant H * K$, there is a dead end $g \in A \wr (H * K)$ of arbitrarily large depth of the form $g = (f, x)$. That is, the position x of the lamplighter in a dead end can be at an arbitrary distance from the identity of $H * K$. This is a difference with the behavior of dead ends of lamplighters over free groups, where the position of the lamplighter for a dead end is necessarily the identity element (Proposition 2.3.3).

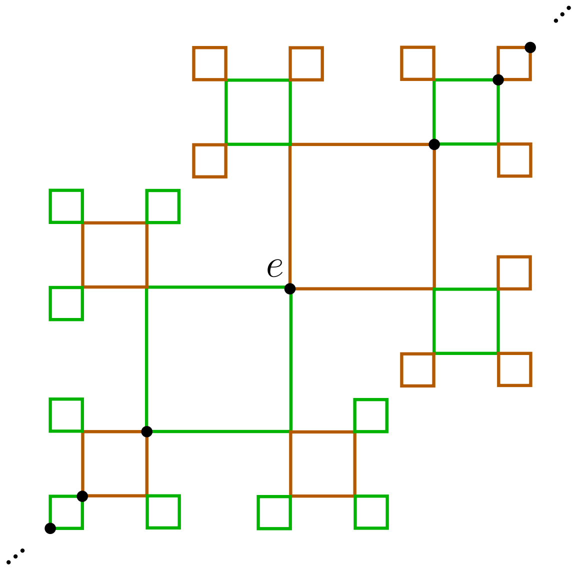


Figure 2.8 – The lamplighter position of a dead end of arbitrary depth of $(A \wr (\mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/4\mathbb{Z}), S_{\text{std}})$ can be any element inside the subgroup $\langle b^2, c^2 \rangle$.

Recall that a finite graph is said to be Hamiltonian-connected if any pair of distinct vertices can be joined by a Hamiltonian path. This is not possible in a bipartite graph, so in that case the strongest possible condition is being Hamiltonian-laceable: having Hamiltonian paths between any two vertices of distinct partite sets of the graph.

Lemma 2.5.5. *Let (G, S_G) be a finite group together with a finite generating set. If* $\mathrm{Cay}(G, S_G)$ *is either Hamiltonian-connected or Hamiltonian-laceable, then*

$$\mathscr{H}(G, S_G) \leq 0.$$

Proof. If $\mathrm{Cay}(G, S_G)$ is Hamiltonian-connected, then $\mathrm{TS}\,(e_G, e_G, G) = |G| + 1$, while for any $g \in G \backslash \{e_G\}$ we have $\mathrm{TS}\,(1_G, g, G) = |G|$. This implies that $\mathscr{H}(G, S_G) = -1$.

Now suppose $\mathrm{Cay}(G, S_G)$ is Hamiltonian-laceable, so that in particular it is bipartite. Write $G = A \cup B$ where A and B form a partition with $e_G \in G$. By considering a Hamiltonian path from $e_G \in A$ to a generator in $S_G \subseteq B$, we see that $\mathrm{TS}\,(e_G, e_G, G) = |G| + 1$.

Now for any $g \in G \backslash \{e_G\}$, we consider two cases. If $g \in B$, then there is a Hamiltonian path from e_G to g so that $\mathrm{TS}\,(e_G, g, G) = |G|$. On the contrary, if $g \in A$ then there is a Hamiltonian path from e_G to a neighbor of g and hence $\mathrm{TS}\,(e_G, g, G) = |G| + 1$. We conclude that $\mathscr{H}(G, S_G) = 0$. □

Corollary 2.5.6. *Suppose that (H, S_H) and (K, S_K) are two finite groups with finite generating sets, which are both either Hamiltonian-connected or Hamiltonian-laceable. Then for any group (A, S_A) with unbounded depth, the lamplighter group $A \wr (H * K)$ with the corresponding standard generating set S_{std} has unbounded depth.*

Proof. Lemma 2.5.5 implies that

$$\mathscr{H}(K, S_K) + \mathscr{H}(H, S_H) \leq 0,$$

which is precisely the negation of Condition (2.3) in Theorem 2.5.3. □

The following corollary characterizes the case of free products of finite abelian groups, generalizing our previous comments about cyclic groups, and covering all possible Cayley graphs. One can think of this corollary as saying that among lamplighters over the free products of two finite abelian groups, the only way to get uniformly bounded depth is for the finite groups forming the base to be sufficiently long cycles.

Corollary 2.5.7. *Suppose that (H, S_H) and (K, S_K) are two finite abelian groups. For any group (A, S_A) with unbounded depth, consider the lamplighter group $A \wr (H * K)$ with the corresponding standard generating set S_{std}. We now list all possible cases for H and K.*

1. *If $|H| = 1$ or $|K| = 1$, then $(A \wr (H * K), S_{\text{std}})$ has unbounded depth.*
2. *If $|H| \in \{2, 3\}$ (resp. $|K| \in \{2, 3\}$), then*
   - *(a) if $\mathrm{Cay}(K, S_K)$ (resp. $\mathrm{Cay}(H, S_H)$) is a cycle of length at least 8, then $(A \wr (H * K), S_{\text{std}})$ has uniformly bounded depth, and*
   - *(b) otherwise $(A \wr (H * K), S_{\text{std}})$ has unbounded depth.*

Now suppose that $|H|, |K| \geq 4$.

3. *If neither $\mathrm{Cay}(H, S_H)$ nor $\mathrm{Cay}(K, S_K)$ are cycles, then $(A \wr (H * K), S_{\text{std}})$ has unbounded depth.*
4. *Suppose that $\mathrm{Cay}(H, S_H)$ is a cycle.*
   - *(a) If $|H| \in \{4, 5\}$, then $(A \wr (H * K), S_{\text{std}})$ has uniformly bounded depth if and only if $\mathrm{Cay}(K, S_K)$ is a cycle of length at least 6.*
   - *(b) If $|H| \in \{6, 7\}$, then $(A \wr (H * K), S_{\text{std}})$ has uniformly bounded depth if and only if $\mathrm{Cay}(K, S_K)$ is a cycle or bipartite.*
   - *(c) If $|H| \geq 8$, then $(A \wr (H * K), S_{\text{std}})$ has uniformly bounded depth.*
5. *An analogous statement to (4) holds when $\mathrm{Cay}(K, S_K)$ is a cycle.*

Proof. 1. If $|H| = 1$ or $|K| = 1$, then $H * K$ is a finite group, and the result follows from Proposition 2.3.1.

2. Suppose that $|H| \in \{2, 3\}$. Then $H \cong \mathbb{Z}/2\mathbb{Z}$ or $H \cong \mathbb{Z}/3\mathbb{Z}$, and in both cases it holds that $\mathscr{H}(H, S_H) = -1$.

   (a) If $\mathrm{Cay}(K, S_K)$ is a cycle of length $\ell \geq 8$, number its vertices cyclically $w_0, \ldots, w_{\ell-1}$ where w_0 is the identity element. Then the vertex indexed by $\lfloor \frac{\ell}{2} \rfloor$ satisfies

$$\mathrm{TS}\left(e_K, w_{\lfloor \frac{\ell}{2} \rfloor}, K\right) \geq \mathrm{TS}\left(e_K, e_K, K\right) + 2,$$

   and so $\mathscr{H}(K, S_K) \geq 2$. This implies that

$$\mathscr{H}(H, S_H) + \mathscr{H}(K, S_K) \geq -1 + 2 = 1,$$

   and hence Condition (2.3) holds.

   (b) In any other case, Proposition 2.2.5 implies that $\mathrm{Cay}(K, S_K)$ is either Hamiltonian-connected or Hamiltonian-laceable. In both cases, Corollary 2.5.6 proves that the corresponding lamplighter group has unbounded depth.

3. If neither $\mathrm{Cay}(H, S_H)$ nor $\mathrm{Cay}(K, S_K)$ are cycles, then both of these graphs are either Hamiltonian-connected or Hamiltonian laceable thanks to Proposition 2.2.5. Then Corollary 2.5.6 implies that $(A \wr (H * K), S_{\mathrm{std}})$ has unbounded depth.

4. Now we suppose that $\mathrm{Cay}(H, S_H)$ is a cycle of length at least 4.

   (a) If $|H| \in \{4, 5\}$, then $\max_{v \in H \setminus \{e_H\}} \left\{\mathrm{TS}\left(e_H, v, H\right)\right\} = \mathrm{TS}\left(e_H, e_H, H\right)$ and hence $\mathscr{H}(H, S_H) = 0$. Then if $\mathrm{Cay}(K, S_K)$ is a cycle of length at least 6 we have

$$\max_{w \in K \setminus \{e_K\}} \left\{\mathrm{TS}\left(e_K, w, K\right)\right\} = \mathrm{TS}\left(e_K, e_K, K\right) + 1,$$

   and in any other case Proposition 2.2.5 together with Lemma 2.5.5 show that

$$\max_{w \in K \setminus \{e_K\}} \left\{\mathrm{TS}\left(e_K, w, K\right)\right\} \leq \mathrm{TS}\left(e_K, e_K, K\right).$$

   In the first case Condition (2.3) in Theorem 2.5.3 is satisfied, while on the second its negation holds.

   (b) If $|H| \in \{6, 7\}$, then $\max_{v \in H \setminus \{e_H\}} \left\{\mathrm{TS}\left(e_H, v, H\right)\right\} = \mathrm{TS}\left(e_H, e_H, H\right) + 1$. If $\mathrm{Cay}(K, S_K)$ is a cycle, it must have length at least 4 and so

$$\max_{w \in K \setminus \{e_K\}} \left\{\mathrm{TS}\left(e_K, w, K\right)\right\} \geq \mathrm{TS}\left(e_K, e_K, K\right).$$

   On the other hand, if $\mathrm{Cay}(K, S_K)$ is not a cycle then

$$\max_{w \in K \setminus \{e_K\}} \left\{\mathrm{TS}\left(e_K, w, K\right)\right\} = \mathrm{TS}\left(e_K, e_K, K\right),$$

if $\mathrm{Cay}(K, S_K)$ is bipartite, and

$$\max_{w \in K \setminus \{e_K\}} \left\{ \mathrm{TS}\left(e_K, w, K\right) \right\} = \mathrm{TS}\left(e_K, e_K, K\right) - 1,$$

otherwise. The first two cases satisfy Condition (2.3) in Theorem 2.5.3 while the third one does not.

(c) If $|H| \geq 8$, then $\max_{v \in H \setminus \{e_H\}} \left\{ \mathrm{TS}\left(e_H, v, H\right) \right\} \geq \mathrm{TS}\left(e_H, e_H, H\right) + 2$. In general, we have that

$$\max_{w \in K \setminus \{e_K\}} \left\{ \mathrm{TS}\left(e_K, w, K\right) \right\} \geq \mathrm{TS}\left(e_K, e_K, K\right) - 1,$$

so that Condition (2.3) in Theorem 2.5.3 is always satisfied.

5. An analogous proof replacing H by K and vice-versa proves the analogous statement to the above.

□

Part II

Random walks and the identification of Poisson boundaries

Chapter 3

Random walks on groups and the Poisson boundary

In this chapter we review the theory of random walks on groups, which is the framework of Chapters 4 and 5 of this thesis. In Section 3.1 we recall classical results of random walks. In Section 3.2 we define the Poisson boundary of a random walk together with other alternative definitions, and explain the associated representation of bounded harmonic functions on the group. Next, in Section 3.3 we comment on the non-triviality and Poisson boundaries and its relation with amenability. Then, in Section 3.4 we introduce entropy theory for random walks on groups, discuss the relation of entropy with speed and volume growth, and state entropy criteria for determining non-triviality and the identification of the Poisson boundary, due to Kaimanovich-Vershik and Kaimanovich. In Section 3.5 we give a summary of families of groups for which there are known complete descriptions of the Poisson boundary, with a detailed discussion of the case of wreath products. Finally, in Section 3.6 we explain the pin-down approximation that has appeared recently in the literature, which has been used to completely describe Poisson boundaries for general classes of measures with finite entropy.

The main references used for this chapter are the books [Lalley, 2023; LyonsPeres, 2021b; Woess, 2000; Woess, 2009; Yadin, 2024], the articles [KaimanovichVershik, 1983; Kaimanovich, 2000; Kaimanovich, 2001] and the surveys [Erschler, 2010; Furman, 2002; Kaimanovich, 2001; Zheng, 2023].

3.1 Random walks on groups

The area of random walks on groups finds its origins at the beginning of the 20th century. The first occurrence of the term "random walk" was in a note by Karl Pearson in [Pearson, 1905], who asked about the distribution at time n of the distance to the origin of a random walk on a plane. Later, Henri Poincaré explained in [PoincaréQuiquet, 1912, Chapitre XVI.215] how the shuffling of a card deck is modeled by the multiplication of randomly chosen elements of a finite permutation group. The first paper that discussed random walks on infinite abelian groups was by George Pólya [Pólya, 1921], who showed that the simple random walk on the d-dimensional lattice is transient if and only if $d \geq 3$.

The origin of the study of random walks on *non-abelian* infinite discrete groups is often attributed to Harry Kesten's PhD Thesis [Kesten, 1959b], who studied the relation between the structure of a group G and the spectrum of the operator on $\ell^2(G)$ defined by the transition probabilities of a symmetric random walk on G. Kesten proved that a group is non-amenable if and only if for a simple random walk, the probability of return to the identity in $2n$ steps decays exponentially fast (i.e. its spectral radius is > 1) [Kesten, 1959a] (see Theorem 1.3.2). In the following decades, the study of random walks on groups gained interest and has been studied using geometric and analytic tools. For a more detailed account of the history of the study of random walks on groups, we refer to the introduction of [Saloff-CosteZheng, 2021].

3.1.1 Basic definitions

Let G be a countable group, and let μ be a probability measure on G. The *support* of μ is defined by $\operatorname{supp}(\mu) \coloneqq \{g \in G \mid \mu(g) > 0\}$. We say that the measure μ is

— *adapted* if $\operatorname{supp}(\mu)$ generates G as a group.

— *non-degenerate* if $\operatorname{supp}(\mu)$ generates G as a semigroup. That is, if $G = \bigcup_{n\geq 0} \operatorname{supp}(\mu)^n$.

We will assume that the probability measures we consider are non-degenerate, unless stated otherwise.

Given probability measures μ_1, μ_2 on G, their *convolution* $\mu_1 * \mu_2$ is defined as the push-forward of the product measure $\mu_1 \times \mu_2$ via the map

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto gh. \end{aligned}$$

Similarly, if (X, ν) is a measure space endowed with a measurable G-action, the convolution $\mu * \nu$ is the push-forward of $\mu \times \nu$ through the map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx. \end{aligned}$$

Equivalently, for every measurable subset $A \subseteq X$ one has $\mu * \nu(A) = \sum_{g\in G} \mu(g)\nu(g^{-1}A)$.

Definition 3.1.1. Fix a probability measure θ on G. The *(right) μ-random walk on G with initial distribution* θ is the Markov chain $\{w_n\}_{n\geq 0}$ with state space G, where the starting point w_0 is distributed according to θ, and with transition probabilities $p(g, h) = \mu(g^{-1}h)$, for $g, h \in G$.

In other words, $w_n = g_0 g_1 \cdots g_n$, $n \geq 1$, where $\{g_i\}_{i\geq 1}$ is a sequence of independent increments with law μ. Note that the law of w_n is given by $\mu_n \coloneqq \theta * \mu^{*n}$, for $n \geq 1$.

The space $G^{\mathbb{Z}_+}$ of elements $\mathbf{w} = \{w_n\}_{n\geq 0}$, endowed with the G-action of pointwise multiplication on the left is called the *space of trajectories* or *path space*. The σ-algebra $\mathcal{A}$ of this space is the one generated by the cylinders $C_g^i \coloneqq \{\mathbf{w} \in G^{\mathbb{Z}_+} \mid w_i = g\}$, for $i \geq 0$ and $g \in G$.

The probability measure $\mathbb{P}_\theta$ on the path space $G^{\mathbb{Z}_+}$ associated with the initial distribution θ is defined as the push-forward of the measure $\theta \otimes \mu^{\mathbb{Z}_+}$ through the map

$$G^{\mathbb{Z}_+} \to G^{\mathbb{Z}_+}$$
$$(g_0, g_1, g_2, \ldots) \mapsto (w_0, w_1, w_2, \ldots) := (g_0, g_0 g_1, g_0 g_1 g_2, \ldots).$$

When $\theta = \delta_g$ for some $g \in G$, we will denote $\mathbb{P}_g := \mathbb{P}_{\delta_g}$. In particular, if $g = e_G$ we will write $\mathbb{P} := \mathbb{P}_{e_G}$. Note that we have $\mathbb{P}_g = g_* \mathbb{P}$ for every $g \in G$, and more generally $\mathbb{P}_\theta = \sum_{g \in G} \theta(g) g_* \mathbb{P}$ for an arbitrary probability measure θ. In this thesis we will work most of the time in the case where the initial distribution θ is the Dirac measure δ_{e_G} concentrated at the identity element e_G.

A very useful tool in the study of random walks, and more generally in the area of dynamical systems, is Kingman's Subadditive Ergodic Theorem [Kingman, 1968]. We state this result below in the setting of the space of increments $(G^{\mathbb{Z}_+}, \mu^{\mathbb{Z}_+})$ endowed with the probability measure-preserving transformation $\sigma : G^{\mathbb{Z}_+} \to G^{\mathbb{Z}_+}$ given by $\sigma(g_1, g_2, g_3, \ldots) := (g_2, g_3, \ldots)$ for every $(g_1, g_2, g_3, \ldots) \in G^{\mathbb{Z}_+}$.

Theorem 3.1.2 (Kingman's Subadditive Ergodic Theorem)**.** *Let G be a group and let μ be a probability measure on G. Suppose that $\{a_n\}_{n \geq 1}$ is a subadditive sequence of non-negative real random variables on the space of increments $(G^{\mathbb{Z}_+}, \mu^{\mathbb{Z}_+})$. That is, for all $n, m \geq 1$ and $\mathbf{g} \in G^{\mathbb{Z}_+}$, it holds that $a_{n+m}(\mathbf{g}) \leq a_n(\mathbf{g}) + a_m \circ \sigma^n(\mathbf{g})$. Additionally, let us suppose that $\mathbb{E}(a_1) < \infty$. Then there is a non-negative constant A such that*

$$\lim_{n \to \infty} \frac{1}{n} a_n(\mathbf{g}) = A, \text{ almost surely.}$$

Furthermore, the convergence also holds in $L^1(G^{\mathbb{Z}_+}, \mu^{\mathbb{Z}_+})$.

The fact that the limit is always a constant is a consequence of the ergodicity of σ on $(G^{\mathbb{Z}_+}, \mu^{\mathbb{Z}_+})$. This result is a common tool in ergodic theory and it has many applications in the theory of random walks on groups. Notably, it is used for proving that the speed of a random walks whose step distribution has a finite first moment is well-defined (Proposition 3.4.12), and for showing that the asymptotic entropy satisfies a Shannon-McMilan-Breiman type property (Theorems 3.4.5 and 3.4.25).

It will also be useful to define the shift in the space of trajectories, which will play a role in the definition of the Poisson boundary of Subsection 3.2.1.

Definition 3.1.3. We define the *time shift* T on the space of trajectories $(G^{\mathbb{Z}_+}, \mathbb{P})$ by

$$T : G^{\mathbb{Z}_+} \to G^{\mathbb{Z}_+}$$
$$(w_0, w_1, w_2, \ldots) \mapsto (w_1, w_2, \ldots).$$

We remark that this is different from the transformation $\sigma : G^{\mathbb{Z}_+} \to G^{\mathbb{Z}_+}$ defined above: whereas the shift σ in the space of increments $(G^{\mathbb{Z}_+}, \mu^{\mathbb{Z}_+})$ is a measure-preserving transformation, the time shift T in the space of trajectories $(G^{\mathbb{Z}_+}, \mathbb{P})$ is not.

Another very useful tool in probability and random walks is the Martingale Convergence Theorem (see for example [Durrett, 2019, Theorem 4.2.11]). Recall that given a probability space $(X, \mathcal{B}, m)$, a *filtration* is family of sub-σ-algebras $\mathcal{F}_n \subseteq \mathcal{B}$, $n \geq 1$, that is increasing in the sense that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for every $n \geq 1$. A sequence $\{M_n\}_{n\geq 1}$ of real-valued random variables on $(X, \mathcal{B}, m)$ is called a *martingale* (resp. sub-martingale) with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1}$ if $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n$ (resp. $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] \geq M_n$) for every $n \geq 1$.

Theorem 3.1.4 (Martingale Convergence Theorem)**.** *Let $(X, \mathcal{B}, m)$ be a probability space and let $\{M_n\}_{n\geq 1}$ be a sub-martingale with respect to a filtration $\{\mathcal{F}_n\}_{n\geq 1}$, such that $\sup_{n\geq 1} \mathbb{E}[M_n \mathbb{1}_{M_n>0}] < \infty$. Then there exists a random variable M_∞ such that $M_n \xrightarrow[n\to\infty]{} M_\infty$ almost surely, and $\mathbb{E}[|M_\infty|] < \infty$.*

In the context of random walks on groups, this theorem is usually applied in the space of trajectories $(G^{\mathbb{Z}_+}, \mathbb{P})$ endowed with the natural filtration associated with the μ-random walk, that is, $\mathcal{F}_n = \sigma(w_1, w_2, \ldots, w_n)$, for $n \geq 1$. The condition $\sup_{n\geq 1} \mathbb{E}[M_n \mathbb{1}_{M_n>0}] < \infty$ holds in particular whenever the sub-martingale $\{M_n\}_{n\geq 1}$ is bounded, in the sense that there is a constant $K \geq 0$ such that $\sup_{n\geq 1} |M_n| \leq K$ almost surely.

3.1.2 Recurrence and transience

For $x, y \in G$ and $n \geq 0$, let us denote by $p^{(n)}(x, y) \coloneqq \mathbb{P}_x(w_n = y)$ the probability that the μ-random walk starting at x is at y after n steps.

Definition 3.1.5. The *Green function* associated with (G, μ) is the power series

$$G(x, y \mid z) \coloneqq \sum_{n=0}^{\infty} p^{(n)}(x, y) z^n, \text{ for } x, y \in G, z \in \mathbb{C}.$$

Let us denote $G(x, y) \coloneqq G(x, y \mid 1)$, for $x, y \in G$, and call this function the *Green kernel* of the μ-random walk on G. Note that

$$G(x, y) = \mathbb{E}_x \left(\sum_{n=0}^{\infty} \mathbb{1}_{\{w_n = y\}} \right)$$

is the expected number of visits to y, when the random walk starts at $w_0 = x$.

Definition 3.1.6. The random walk (G, μ) is called *recurrent* if $G(x, y) = \infty$ for some (equivalently, every) $x, y \in G$. Otherwise, it is called *transient*.

Historically, the first result on the asymptotic properties of random walks on infinite groups was by Pólya in 1921, who studied the recurrence of simple random walks on the d-dimensional lattice.

Theorem 3.1.7 ([Pólya, 1921])**.** *The simple random walk on $\mathbb{Z}^d$ is recurrent if and only if $d \leq 2$.*

The *spectral radius* of the μ-random walk on G is defined as

$$\rho \coloneqq \limsup_{n\to\infty} \left(p^{(n)}(x, y)\right)^{1/n} \in (0, 1], \text{ for } x, y \in G,$$

and it can be proved that its value does not depend on the choices of x and y. The following lemma relates the spectral radius of the random walk with the Green function.

Lemma 3.1.8. *For all $x, y \in G$, the radius of convergence of the power series $G(x, y \mid z)$ is $\mathfrak{r} := 1/\rho$.*

Note that if μ is non-degenerate and (G, μ) is recurrent, then $G(x, y \mid 1)$ diverges for every $x, y \in G$. This implies that the radius of convergence of the Green function is 1, and hence that $\rho = 1$.

Kesten's criterion [Kesten, 1959a] states that $\rho = 1$ if and only if G is amenable (see Theorem 1.3.2). Thus, due to the above observation, any group that admits a recurrent random walk is amenable. Kesten asked in [Kesten, 1967] whether it is possible to characterize all finitely generated groups that admit a non-degenerate recurrent random walk. This question was answered by Varopoulos.

Theorem 3.1.9 ([Varopoulos, 1986])**.** *Suppose that G is infinite, finitely generated, and that (G, μ) is recurrent for some non-degenerate probability measure μ. Then G contains a finite index subgroup isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$.*

Varopoulos' proof uses the fact that lower bounds for the volume growth of a group give upper bounds for the decay of the return probability to the identity. More precisely, let γ be the growth function of G with respect to some generating set S and let μ be a symmetric non-degenerate probability measure on G, and recall the asymptotic relations of functions $\preccurlyeq$ and $\sim$ introduced in Section 1.1.2. Suppose that $\gamma(n) \succcurlyeq n^d$ for some $d \geq 1$. Then there exists a constant $C > 0$ such that $\mu^{\star 2n}(e_G) \leq C n^{-d/2}$. Varopoulos also uses Gromov's theorem of polynomial growth to argue that if the growth function γ of G satisfies $\gamma(n) \preccurlyeq n^2$ then G must be virtually $\mathbb{Z}$ or $\mathbb{Z}^2$. We refer to [Ancona, 1990, Chapitre III.4] and [Woess, 2000, Chapter I.3.B] for an exposition of the proof of Varopoulos' Theorem.

3.2 The Poisson boundary

A classical result in complex and harmonic analysis is the *Poisson integral representation formula*, which provides a correspondence between bounded harmonic functions on the unit disk $\mathbb{D} \subseteq \mathbb{C}$ and bounded measurable functions on the circle $S^1 = \partial\mathbb{D}$. More precisely, consider the space $H^\infty(\mathbb{D})$ of functions $u : \mathbb{D} \to \mathbb{R}$ such that $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (i.e. u is harmonic), and $\sup_{x+iy \in \mathbb{D}} |u(x, y)| < \infty$. Denote by $L^\infty(S^1)$ the space of bounded measurable function on the unit circle S^1. Each bounded harmonic function $u \in H^\infty(\mathbb{D})$ has an extension to S^1. Indeed, one can define for each $\xi \in S^1$ the value $u(\xi) := \lim_{r \to 1^-} u(r\xi)$. Alternatively, one can consider a "hyperbolic Brownian motion" $\{B_t\}_{t \geq 0}$ on the disk $\mathbb{D}$ (seen as the Poincaré model of the hyperbolic plane), which satisfies that $B_\infty := \lim_{t \to \infty} B_t$ exists almost surely and is an element of S^1. Then one can define for every $\xi \in S^1$ the value $u(\xi) := \lim_{t \to \infty} \mathbb{E}[u(B_t) \mid B_\infty = \xi]$. Reciprocally, every function $F \in L^\infty(S^1)$ uniquely determines a bounded harmonic function $u \in H^\infty(\mathbb{D})$ which extends to F as follows. Let us define the *Poisson kernel* $P_r(\theta) := \frac{1-r^2}{1-2r\cos(\theta)+r^2}$, for $0 \le r < 1$ and $-\pi \le \theta < \pi$. Then for every $F \in L^\infty(S^1)$, the function $u : \mathbb{D} \to \mathbb{R}$ defined by

$$u(re^{i\theta}) \coloneqq \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})P_r(\theta - t)\,\mathrm{d}t \text{ for } 0 \le r < 1 \text{ and } -\pi \le \theta < \pi, \tag{3.1}$$

belongs to $H^\infty(\mathbb{D})$, and its extension to S^1 coincides with F. This is known as the *Poisson integral representation formula*. Denote by λ the Lebesgue probability measure on S^1, and for every $g \in \mathrm{SL}(2,\mathbb{R})$ consider the push-forward measure $g_*\lambda$. Then it can be shown that Equation (3.1) takes the form

$$u(z) = \int_{S^1} F(\xi)\,\mathrm{d}g_*\lambda(\xi), \tag{3.2}$$

where $g \in \mathrm{SL}(2,\mathbb{R})$ satisfies $g(0) = z$.

Given a countable group G with a probability measure μ, its Poisson boundary is a probability space (B,ν) which describes the asymptotic behavior of the μ-random walk on G, and there is a correspondence between measurable bounded functions on (B,ν) and bounded μ-harmonic functions on G, in analogy with Equation (3.2). In particular, the Poisson boundary (B,ν) is trivial (i.e. is the space with one point) if and only if every bounded μ-harmonic function on G is constant. The origins of the concept of the Poisson boundary go back to [Blackwell, 1955; Doob, 1959; Feller, 1956] for general Markov chains, and to [DynkinMaljutov, 1961] and [Furstenberg, 1963; Furstenberg, 1967; Furstenberg, 1971; Furstenberg, 1973] more precisely for random walks on groups. Due to Harry Furstenberg's contribution to this theory, some authors use the name "Poisson-Furstenberg boundary", e.g. in [BartholdiErschler, 2017; Erschler, 2010; Vershik, 2000a].

The theory of Poisson boundaries has diverse applications in group theory. One such instance is in determining the amenability of a group, which is equivalent to the triviality of the Poisson boundary for at least one non-degenerate measure [Azencott, 1970; Furstenberg, 1967; KaimanovichVershik, 1983; Rosenblatt, 1981] (see Theorem 3.3.1). This has been used to prove the amenability of various classes of groups, notably some groups acting on rooted trees such as the Basilica group, and we refer Section 1.3 for a more detailed discussion on this. The Poisson boundary can also be used to study the growth of finitely generated groups. More precisely, it is a consequence of the entropy criterion [Derriennic, 1980; KaimanovichVershik, 1983] (see Section 3.4) that probability measures with a finite first moment in a group of subexponential growth have a trivial Poisson boundary. In contrast, it is a consequence of [FrischHartmanTamuzVahidi Ferdowsi, 2019] that every finitely generated group of intermediate growth admits probability measures with a non-trivial Poisson boundary. Examples of such measures in specific groups of intermediate growth and with a control over their tail decay have been constructed in [Erschler, 2004a; ErschlerZheng, 2020]. It can be shown that a known tail decay of probability measures with non-trivial boundary leads to upper estimates for the volume growth of the group, and in particular this was used in [ErschlerZheng, 2020] to give near-optimal upper bounds for the volume growth of Grigorchuk's group.

Other applications of the theory of Poisson boundaries can be found in Furstenberg's approach to superrigidity results [Furstenberg, 1971], and the appearance of stationary measures on geometric boundaries in the proof of monotonicity of the hyperbolic volume for finite volume non-compact manifolds [Thurston, 1977] (see also [Gromov, 1981a, Section 6] and the remark following Lemma 6.4.5 in [Thurston, 2022]). The Poisson boundary of lattices in semisimple Lie groups is used in [EskinMatheus, 2015] to provide a simplicity criterion for the Lyapunov spectrum of the Kontsevich-Zorich cocycle over Teichmüller curves, that does not use combinatorial codings of the Teichmüller flow. This is in contrast with previously known simplicity criteria, which do rely on combinatorial codings. We refer to Carlos Matheus' blog post [Matheus, 2012] for a more detailed explanation on how the Poisson boundary is used for this result. The theory of Poisson boundaries also appears in the proof of Bader-Shalom's normal subgroup theorem [BaderShalom, 2006], where it is used to establish amenability.

The study of spaces of harmonic functions on groups also has applications beyond the case of bounded functions (which corresponds to the Poisson boundary). For example, in Bruce Kleiner's proof of Gromov's theorem of polynomial growth [Kleiner, 2010], it is shown that if G is an infinite finitely generated group of polynomial growth and μ is the uniform probability measure on some arbitrary finite generating set of G, the vector space of μ-harmonic functions on G that have polynomial growth of at most some fixed degree d is finite-dimensional. This is then used to prove that G admits finite-dimensional representations with an infinite image. In contrast, the original proofs of Gromov's theorem show the existence of such representations by using the Montgomery-Zippin-Yamabe structure theory of locally compact groups, which is related to Hilbert's fifth problem.

3.2.1 Definition of the Poisson boundary

Consider a countable group G endowed with a probability measure μ. Let us denote by m the counting measure on G, and consider the associated σ-finite measure $\mathbb{P}_m$ on the space of trajectories $G^{\mathbb{Z}_+}$ which corresponds to the μ-random walk on G with initial distribution m. Then for any choice of initial distribution θ, the measure $\mathbb{P}_\theta$ is absolutely continuous with respect to $\mathbb{P}_m$, and they are equivalent (i.e. they have the same null sets) if $\mathrm{supp}(\theta) = G$.

For any initial distribution θ, the push-forward of $\mathbb{P}_\theta$ through the time shift T on the space of trajectories (Definition 3.1.3) satisfies $T_*\mathbb{P}_\theta = \mathbb{P}_{\theta\mu}$. In particular if $\mathrm{supp}(\theta) = G$, the measure $\mathbb{P}_\theta$ is quasi-invariant with respect to T, in the sense that $T_*\mathbb{P}_\theta$ is absolutely continuous with respect to $\mathbb{P}_\theta$. Furthermore, for the counting measure m on G it holds that $\mathbb{P}_m$ is T-invariant.

The space of sample paths $(G^{\mathbb{Z}_+}, \overline{\mathcal{A}}, \mathbb{P}_m)$, where $\overline{\mathcal{A}}$ is the completion of the σ-algebra generated by the cylinder sets, is a Lebesgue space and the action of the time shift T preserves the measure $\mathbb{P}_m$. We can thus study this space using Rokhlin's theory of Lebesgue spaces [Rohlin, 1949; Rohlin, 1952] (see also [Rohlin, 1967] and [CornfeldFominG, 1982, Appendix 1]), which in this context establishes a one-to-one correspondence between complete sub-σ-algebras of $\overline{\mathcal{A}}$ and measurable partitions of $(G^{\mathbb{Z}_+}, \overline{\mathcal{A}}, \mathbb{P}_m)$.

Let us introduce a measurable partition of $(G^{\mathbb{Z}_+}, \overline{\mathcal{A}}, \mathbb{P}_m)$ via the following equivalence relation.

Definition 3.2.1. We define the *asymptotic equivalence* relation $\sim$ on the space of trajectories by saying that for every $\mathbf{w}, \mathbf{w}' \in G^{\mathbb{Z}_+}$, $\mathbf{w} \sim \mathbf{w}'$ if and only if there exist $k, N \geq 0$ such that $w_n = w'_{n+k}$ for all $n \geq N$.

Alternatively, we have $\mathbf{w} \sim \mathbf{w}'$ if and only if there exist $i, j \geq 0$ such that $T^i(\mathbf{w}) = T^j(\mathbf{w}')$.

Denote by $\mathcal{I}$ the complete σ-algebra of measurable ($\mathbb{P}_m$-mod 0) T-invariant subsets of the space of trajectories $(G^{\mathbb{Z}_+}, \mathbb{P}_m)$. That is, $\mathcal{I}$ is the σ-algebra of all measurable subsets of $G^{\mathbb{Z}_+}$ which are unions ($\mathbb{P}_m$-mod 0) of the equivalence classes of $\sim$. The σ-algebra $\mathcal{I}$ is called the *measurable hull* of the equivalence relation $\sim$. Rokhlin's correspondence associates with the complete sub-σ-algebra $\mathcal{I}$ of $\overline{\mathcal{A}}$ a unique measurable partition η of the space of sample paths $G^{\mathbb{Z}_+}$, called the *Poisson partition*, which is well-defined up to $\mathbb{P}_m$-null sets. The atoms of η are the ergodic components of the shift map T. The *Poisson boundary* $\partial_\mu G$ is the quotient space $G^{\mathbb{Z}_+}/\eta$ induced by the Poisson partition η, and it is endowed with the image of the σ-algebra $\mathcal{I}$ through the quotient map $\mathbf{bnd} : G^{\mathbb{Z}_+} \to \partial_\mu G$. The measure $\nu := \mathbf{bnd}_*\mathbb{P}$ is called the *harmonic measure on* $\partial_\mu G$ associated with the initial distribution $\theta = \delta_{e_G}$ in the space of trajectories $G^{\mathbb{Z}_+}$.

Definition 3.2.2. The probability space $(\partial_\mu G, \nu)$ is called the **Poisson boundary** of (G, μ).

Note that we have

$$\nu = \mathbf{bnd}_*\mathbb{P} = \mathbf{bnd}_* T_* \mathbb{P} = \mathbf{bnd}_* \mathbb{P}_\mu = \mu * \mathbf{bnd}_*\mathbb{P} = \mu * \nu.$$

In other words, the measure ν on $\partial_\mu G$ is μ*-stationary.* We remark that this is not the case for the push-forward measure on $\partial_\mu G$ induced by other initial distributions.

In the remaining of this section, we define harmonic functions and explain the correspondence between measurable bounded functions on the Poisson boundary of (G, μ) and bounded μ-harmonic functions on G (see Subsection 3.2.3). We also present alternative definitions of the Poisson boundary via the tail σ-algebra (Subsection 3.2.4) and the Martin boundary (Subsection 3.2.5). There are other definitions of the Poisson boundary that we do not mention in further detail in this thesis, such as Furstenberg's original definition [Furstenberg, 1963, Definition 5.4] (see also [Furstenberg, 1971, Section 3.5] and [Azencott, 1970, Définition I.2]) via the Gelfand-Naimark theorem and Riesz's representation theorem, a definition in terms of the Mackey range over the shift map in the space of increments [Zimmer, 1978, Theorem 5.1], a definition using ideals in the group algebra of G [Willis, 1990, Theorem 2.1], and a definition in terms of topological dynamics [DokkenEllis, 1990, Section 1 and Remarks 2.16].

3.2.2 Harmonic functions on groups

From now on and throughout the rest of the chapter, we will consider G a countable group and μ a probability measure on G. We start by defining μ-harmonic functions on G as those that satisfy the mean value property.

Definition 3.2.3. A function $f : G \to \mathbb{R}$ is called μ*-harmonic* if

$$f(g) = \sum_{h \in G} f(gh)\mu(h), \text{ for every } g \in G. \tag{3.3}$$

We denote by $H^\infty(G, \mu)$ the space of μ-harmonic functions $f : G \to \mathbb{R}$ that are bounded, in the sense that $\sup_{g \in G} |f(g)| < \infty$.

Equivalently, Equation (3.3) can be written as $f(g) = \mathbb{E}_g[f(w_1)]$ for every $g \in G$, where $\mathbb{E}_g[\cdot]$ denotes the expectation with respect to the probability measure $\mathbb{P}_g$, which corresponds to the μ-random walk on G that starts at $w_0 = g$. Note that for any group G and any probability measure μ, the constant functions are bounded μ-harmonic functions.

The connection between μ-harmonic functions and the μ-random walk on G is the following. Recall that we denote $\{w_n\}_{n\geq 0}$ the μ-random walk on G (Definition 3.1.1).

Lemma 3.2.4. *For every μ-harmonic function $h : G \to \mathbb{R}$ and any $g \in G$ the sequence of random variables $\{h(gw_n)\}_{n\geq 0}$ is a martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(w_0, w_1, \dots, w_n)$, $n \geq 0$. In consequence, if $h \in H^\infty(G,\mu)$ then for almost every $\mathbf{w} = (w_0, w_1, w_2, \dots) \in (G^{\mathbb{Z}_+}, \mathbb{P})$ and every $g \in G$ the limit $\lim_{n\to\infty} h(gw_n)$ exists.*

Proof. Let $h \in H^\infty(G,\mu)$ be an arbitrary bounded μ-harmonic function, and denote $\mathcal{F}_n = \sigma(w_0, w_1, \dots, w_n)$, $n \geq 0$, the natural filtration associated with the μ-random walk on G. Then $h_n(x) \coloneqq h(x_0 x_1 \cdots x_n)$ is $\mathcal{F}_n$-measurable. Note that

$$\begin{aligned} \mathbb{E}(h_{n+1} \mid \mathcal{F}_n) &= \mathbb{E}(h(x_0 x_1 \cdots x_n x_{n+1}) \mid X_1, \dots, X_n) \\ &= \int h(x_0 x_1 \cdots x_n z) \, \mathrm{d}\mu(z) \\ &= h(x_0 \cdots x_n) = f_n(x). \end{aligned}$$

This proves the first part of the lemma. For the second statement of the lemma, note that $\|h_n\|_\infty \leq \|h\|_\infty$ and hence the martingale convergence theorem (Theorem 3.1.4) guarantees the existence of the limit $\lim_{n\to\infty} h(gw_n)$ for $\mathbb{P}$-almost every $\mathbf{w} = (w_0, w_1, w_2, \dots) \in G^{\mathbb{Z}_+}$. □

Example 3.2.5. Let us consider $G = \mathbb{Z}$ and $\mu = p\delta_1 + (1-p)\delta_{-1}$, for some parameter $0 < p < 1$. Then the μ-random walk on $\mathbb{Z}$ is the random walk on an infinite line, which at each instant moves to the left with probability p and to the right with probability $1 - p$.

A function $f : \mathbb{Z} \to \mathbb{R}$ is μ-harmonic if and only if it satisfies $f(n) = pf(n+1)+(1-p)f(n-1)$ for every $n \in \mathbb{Z}$. If $p = 1/2$, then one can see that this recurrence relation has a general solution of the form $f(n) = a + bn$, $n \in \mathbb{Z}$, where $a, b \in \mathbb{R}$ are constants. When $p \neq 1/2$ the recurrence relation can be solved using the characteristic roots method to obtain the general solution $f(n) = a + b\left(\frac{1-p}{p}\right)^n$, $n \in \mathbb{Z}$, where $a, b \in \mathbb{R}$ are constants. In both cases, we see that $H^\infty(\mathbb{Z},\mu)$ is the space of constant functions.

Example 3.2.6. Consider the free group $G = F_k = F(x_1, x_2, \dots, x_k)$ on $k \geq 2$ generators, and let $\mu \coloneqq \frac{1}{2k}\sum_{i=1}^k(\delta_{x_i} + \delta_{x_i^{-1}})$ be the uniform probability measure on the standard generating set. The μ-random walk on F_k can be identified with the simple random walk on the infinite regular tree of valence $2k$.

A function $f : F_k \to \mathbb{R}$ is μ-harmonic if and only if for every $g \in F_k$,

$$f(g) = \frac{1}{2k}\sum_{i=1}^k \left(f(gx_i) + f(gx_i^{-1})\right).$$

In this case, we will see that it is possible to find non-constant μ-harmonic functions. Indeed, fix arbitrary $j \in \{1, 2, \dots, k\}$ and $\varepsilon \in \{1, -1\}$, and let $A_j \subseteq F_k$ be the set of elements that start with x_j^ε, when viewed as a reduced word in the alphabet $\{x_1, x_2, \dots, x_k\}^{\pm 1}$. Define the function $f(n) : F_k \to \mathbb{R}$ by

$$f(g) = \begin{cases} (2k-1)^{-|g|}, \text{ if } g \in F_k \backslash A_j, \text{ and} \\ 2k - (2k-1)^{1-|g|}, \text{ if } g \in A_j. \end{cases}$$

Here $|\cdot|$ denotes the word length on F_k. The function f is bounded, and we will now see that it is μ-harmonic by looking at the following four cases.

Case 1: Consider $g \in A_j$, with $g \neq x_j^\varepsilon$. Then for every $i \in \{1, \dots, k\}$ we have $gx_i, gx_i^{-1} \in A_j$, and hence

$$\begin{aligned}
\frac{1}{2k}\sum_{i=1}^{k}\Big(f(gx_i) + f(gx_i^{-1})\Big) &= \frac{1}{2k}\sum_{i=1}^{k}\Big(2k - (2k-1)^{1-|gx_i|} + 2k - (2k-1)^{1-|gx_i^{-1}|}\Big) \\
&= \frac{1}{2k}\Big(2k - (2k-1)^{2-|g|} + (2k-1)(2k - (2k-1)^{-|g|})\Big) \\
&= 2k - (2k-1)^{1-|g|} \\
&= f(g).
\end{aligned}$$

Case 2: Now let $g = x_j^\varepsilon$, so that $gx_j^{1-\varepsilon} = e_{F_k}$, $x_j^\varepsilon \in A_j$ and $gx_i, gx_i^{-1} \in A_j$ for every $i \in \{1, 2 \dots, k\}$ with $i \neq j$. Then

$$\frac{1}{2k}\sum_{i=1}^{k}\Big(f(gx_i) + f(gx_i^{-1})\Big) = \frac{1}{2k}\Big((2k-1)^0 + (2k-1)(2k - (2k-1)^{-1})\Big) = 2k - 1 = f(g).$$

Case 3: Consider $g = e_{F_k}$, so that $gx_j^\varepsilon = x_j^\varepsilon \in A_j$, $gx_j^{1-\varepsilon} \in F_k \backslash A_j$ and $gx_i, gx_i^{-1} \in F_k \backslash A_j$ for every $i \in \{1, 2 \dots, k\}$ with $i \neq j$. Now we have

$$\frac{1}{2k}\sum_{i=1}^{k}\Big(f(x_i) + f(x_i^{-1})\Big) = \frac{1}{2k}\Big((2k-1) + (2k-1)(2k-1)^{-1}\Big) = 1 = f(g).$$

Case 4: Finally, let $g \in F_k \backslash A_j$ with $g \neq e_{F_k}$. Then $gx_i, gx_i^{-1} \in F_k \backslash A_j$ for each $i \in \{1, 2, \dots, k\}$, and we have

$$\begin{aligned}
\frac{1}{2k}\sum_{i=1}^{k}\Big(f(x_i) + f(x_i^{-1})\Big) &= \frac{1}{2k}\Big((2k-1)(2k-1)^{-|g|-1} + (2k-1)^{-|g|+1}\Big) \\
&= (2k-1)^{-|g|} \\
&= f(g).
\end{aligned}$$

We conclude that f is a bounded and non-constant μ-harmonic function on F_k.

Definition 3.2.7. Let G be a group and μ a non-degenerate probability measure on G. We say that (G, μ) has the *Liouville property* if $H^\infty(G, \mu)$ is the space of constant functions on G.

We will see below that the space of bounded μ-harmonic functions is closely related to the asymptotic behavior of the μ-random walk on G (see Definition 3.2.16 and Theorem 3.2.17). A first instance of this relation is contained in the next proposition.

Proposition 3.2.8. *Let G be a countable group and μ a probability measure on G. Suppose that the μ-random walk on G is recurrent. Then (G, μ) has the Liouville property.*

Proof. Consider an arbitrary $f \in H^\infty(G, \mu)$. Then, thanks to Lemma 3.2.4, $\{f(w_n)\}_{n \geq 0}$ is a bounded martingale. It follows that there exists a random variable F such that $\mathbb{P}$-almost surely $F = \lim_{n\to\infty} f(w_n)$.

It suffices to prove that for every $g \in G$, we have $f(g) = f(e_G)$. Let $g \in G$ be arbitrary, and looking for a contradiction let us suppose that $f(e_G) < f(g)$. Note that since the μ-random walk is recurrent, we have that $\mathbb{P}$-almost surely $w_i = e_G$ and $w_j = g$ for infinitely many $i, j \in \mathbb{N}$. But then we have

$$F = \liminf_{n\to\infty} f(w_n) \leq f(e_G) < f(g) \leq \limsup_{n\to\infty} f(w_n) = F.$$

This is a contradiction. In a analogous way, if $f(e_G) > f(g)$ then we arrive at a contradiction.

We conclude that $f(e_G) = f(g)$ for every $g \in G$, so that f must be constant. □

In Example 3.2.5 we saw that if μ is a probability measure on $\mathbb{Z}$ supported on the set $\{1, -1\}$, then every bounded μ-harmonic functions on $\mathbb{Z}$ must be constant. This is a particular case of the following general result.

Theorem 3.2.9 ([Blackwell, 1955; ChoquetDeny, 1960; DoobSnellWilliamson, 1960])**.** *Let μ be an adapted measure on an abelian group G. Then (G, μ) has the Liouville property.*

This result is commonly referred to as the Choquet-Deny Theorem. Groups in which every adapted measure has the Liouville property are called *Choquet-Deny groups*. Recently, a characterization of countable Choquet-Deny groups as precisely the countable hyper-FC-central groups has been obtained by [FrischHartmanTamuzVahidi Ferdowsi, 2019]. We discuss this result in further detail in Subsection 3.3.1.

3.2.3 The Poisson formula for groups

Definition 3.2.10. A probability space (X, λ) is called:

1. a *measure G-space* if X is endowed with a measurable action of G, and
2. a *(G, μ)-space* if it is a measure G-space and in addition the measure λ satisfies $\lambda = \mu * \lambda$. In such a case, we say that λ is *μ-stationary.*

That is, λ is μ-stationary if and only if for every measurable subset $A \subseteq X$, we have $\lambda(A) = \sum_{g \in G} \mu(g)\lambda(g^{-1}A)$.

Whenever G acts via probability measure preserving transformations on a probability space (X, λ), then λ is invariant under the action of G and is thus a μ-stationary measure. In general, the μ-stationary measures we will consider will not be G-invariant (see Remark 3.2.13 below).

The following proposition guarantees that any continuous action of G on a compact space admits at least one stationary measure.

Proposition 3.2.11. *Let μ a probability measure on G. For any continuous action of G on a compact Hausdorff metrizable space M, Then the space of μ-stationary measures on M*

$$\mathrm{Stat}(\mu) := \{\nu \in \mathrm{Prob}(M) \mid \nu = \mu * \nu\}.$$

is a non-empty, convex and compact subset, with respect to the weak-topology.*

Proof. Note that since M is compact, it follows that $\mathrm{Prob}(M)$ is weak*-compact. Additionally, $\mathrm{Prob}(M)$ is a convex set, and the map $\mathrm{Prob}(M) \to \mathrm{Prob}(M)$ which maps $\lambda \mapsto \mu * \lambda$ is weak*-continuous and affine. Then, it follows from the Markov-Kakutani fixed point theorem that there exists $\lambda \in \mathrm{Prob}(M)$ such that $\mu * \lambda = \lambda$. This implies that $\mathrm{Stat}(\mu) \neq 0$. □

The following gives a way to construct μ-harmonic functions on G from a given (G, μ)-space.

Proposition 3.2.12. *Let (X, λ) be a (G, μ)-space. Then the following map is well-defined:*

$$\begin{aligned} \Phi : L^\infty(X, \lambda) &\to H^\infty(G, \mu) \\ F &\mapsto \Phi(F)(g) := \int_X F(x)\, \mathrm{d}g_*\lambda(x) = \int_X F(gx)\, \mathrm{d}\lambda(x). \end{aligned} \tag{3.4}$$

We call Φ the Poisson transform *associated with (X, λ).*

Proof. We need to check that for any $F \in L^\infty(X, \lambda)$, its image $\Phi(F)$ is bounded and μ-harmonic. For every $F \in L^\infty(X, \lambda)$, it holds that

$$\|\Phi(F)\|_\infty = \sup_{g \in G} \left| \int_X f(gx)\, \mathrm{d}\lambda(x) \right| \leq \|F\|_\infty,$$

so that $\Phi(F)$ is bounded. Let us now check that it is μ-harmonic. Indeed, for every $g \in G$,

$$\begin{aligned} \sum_{h \in H} \Phi(F)(gh)\mu(h) &= \sum_{h \in H} \int_X F(x)\ \mu(h)\, \mathrm{d}(gh)_*\lambda(x) \\ &= \int_X F(x) \ \sum_{h \in H} \mu(h)\, \mathrm{d}(gh)_*\lambda(x) \\ &= \int_X F(x)\, \mathrm{d}g_*\lambda(x) \\ &= \Phi(F)(g). \end{aligned}$$

Above, we used the fact that λ is μ-stationary. □

Remark 3.2.13. Let (X, λ) be a (G, μ)-space. Then the image of the Poisson transform (Equation (3.4)) consists of only constant functions if and only if λ is a G-invariant measure. That is, $g_*\lambda = \lambda$ for every $g \in G$.

Corollary 3.2.14. *Suppose that G is non-amenable and let μ be a non-degenerate probability measure on G. Then $H^\infty(G, \mu)$ contains non-constant functions.*

Proof. Recall from Proposition 1.3.4 group is amenable if and only if for every continuous action of the group on a compact space, there exists an invariant probability measure. Hence, since G is non-amenable there is a continuous action of G on a compact space X without G-invariant probability measures. Furthermore, thanks to Proposition 3.2.11, the space X admits a μ-stationary probability measure λ. Then thanks to Remark 3.2.13, the image of the Poisson-Furstenberg transform contains non-constant bounded μ-harmonic functions. □

The previous corollary is due to [Azencott, 1970, Proposition II.1] and [Furstenberg, 1973]; see also the paragraph after Theorem 3.3.1 below.

Remark 3.2.15. An alternative proof of Corollary 3.2.14 is presented in [KaimanovichVershik, 1983, Theorem 4.2]. There, it is shown that the fact that $H^\infty(G, \mu)$ consists only of constants functions (which is equivalent to the triviality of the Poisson boundary) implies that the convolution powers μ^{*n}, $n \geq 1$, converge strongly to a left-invariant mean on G, which shows that G must be amenable (see Definition 1.3.1).

Definition 3.2.16. A measure G-space (X, λ) is said to be a *μ-boundary* if there exists a measurable map $G^{\mathbb{Z}_+} \to X$ which is shift-invariant. If, in addition, the associated Poisson-Furstenberg transform (Equation (3.4)) is an isomorphism, then it is said to be a *model of the Poisson boundary* of (G, μ).

Recall the definition of the Poisson boundary $(\partial_\mu G, \nu)$ from Subsection 3.2.1.

Theorem 3.2.17. *The space $(\partial_\mu G, \nu)$ from Definition 3.2.2 is a model of the Poisson boundary of (G, μ), in the sense of Definition 3.2.16.*

Proof. We already know that $(\partial_\mu G, \nu)$ is a μ-boundary. In order to prove that it is a model of the Poisson boundary, let us show that the Poisson transform is an isomorphism.

For an arbitrary $f \in H^\infty(G, \mu)$, let us define $F \in L^\infty(\partial_\mu G, \nu)$ by

$$F\left(\mathbf{bnd}(\mathbf{w})\right) \coloneqq \lim_{n\to\infty} f(w_n), \text{ for } \mathbf{w} = (w_0, w_1, \ldots) \in G^{\mathbb{Z}_+},$$

which is $\mathbb{P}$-almost surely well-defined thanks to Lemma 3.2.4. Then, for every $g \in G$ we have

$$\begin{aligned}
\Phi(F)(g) &= \int_{\partial_\mu G} F(gx)\,\mathrm{d}\nu(x) \\
&= \int_{\partial_\mu G} F(gx)\,\mathrm{d}(\mathbf{bnd}_*\mathbb{P})(x) \\
&= \int_{G^{\mathbb{Z}_+}} F(g\mathbf{bnd}(\mathbf{w}))\,\mathrm{d}\mathbb{P}(\mathbf{w}) \\
&= \int_{G^{\mathbb{Z}_+}} F(\mathbf{bnd}(g\mathbf{w}))\,\mathrm{d}\mathbb{P}(\mathbf{w}) \\
&= \int_{G^{\mathbb{Z}_+}} \lim_{n\to\infty} f(gw_n)\,\mathrm{d}\mathbb{P}(\mathbf{w}) \\
&= \lim_{n\to\infty} \int_{G^{\mathbb{Z}_+}} f(gw_n)\,\mathrm{d}\mathbb{P}(\mathbf{w}) \\
&= f(g),
\end{aligned}$$

where we used the dominated convergence theorem and the fact that since f is μ-harmonic, it is also μ^{*n}-harmonic. We conclude that $\Phi(f) = f$, which finishes the proof. □

The Poisson boundary of (G, μ) is the maximal μ-boundary, in the sense that it satisfies the following universal property.

Proposition 3.2.18. *For any μ-boundary (X, λ) endowed with its boundary map $\varphi : (G^{\mathbb{Z}_+}, \mathbb{P}) \to (X, \lambda)$, there exists a G-equivariant map $\pi : (\partial_\mu G, \nu) \to (X, \lambda)$ such that $\varphi = \pi \circ \mathbf{bnd}$.*

In this thesis, we will work with Definition 3.2.2 for the Poisson boundary of a random walk on a group. In what remains of this section we present other alternative definitions.

3.2.4 The tail boundary

The *tail boundary* of the random walk (G, μ) is obtained by replacing the asymptotic equivalence relation (Definition 3.2.1) with the following.

Definition 3.2.19. Define the *tail equivalence* $\sim_{\text{tail}}$ relation on the space of trajectories by saying that for every $\mathbf{w}, \mathbf{w}' \in G^{\mathbb{Z}_+}$, $\mathbf{w} \sim_{\text{tail}} \mathbf{w}'$ if and only if there exist $N \geq 0$ such that $w_n = w'_n$ for all $n \geq N$.

The difference with Definition 3.2.1 is that now we do not allow for a shift in the trajectory. The *tail σ-algebra* $\mathcal{T}$ is the σ-algebra formed by measurable unions of equivalence classes of $\sim_{\text{tail}}$ mod 0. Alternatively, we can write $\mathcal{T} = \bigcap_{n \geq 0} \sigma(w_n, w_{n+1}, w_{n+2}, \ldots)$.

Any two trajectories that are tail equivalent are also asymptotic equivalent, and hence the tail boundary contains the stationary boundary. For general Markov chains, it is possible that this inclusion is strict; see [BlackwellFreedman, 1964, Example 2] and [Kaimanovich, 1992, Section 4]. For random walks on groups both boundaries coincide up to $\mathbb{P}$-null sets [Derriennic, 1986, Appendice 2], [Kaimanovich, 1992, Theorem 4.5].

3.2.5 The Martin boundary

The Martin boundary is a topological space which provides a complete description of the space of positive (not necessarily bounded) μ-harmonic functions on G. This is in contrast with the Poisson boundary, which is defined in probabilistic terms. Below we define the Martin boundary, and explain how it can be endowed with an appropriate probability measure such that the corresponding measure space is isomorphic to the Poisson boundary of (G, μ). We refer to [Sawyer, 1997] and [Kaimanovich, 1996, Section 1] and [Woess, 2000, Chapter IV.24] and the references therein for more details on the Martin boundary and the proofs of the results mentioned below.

Recall from Definition 3.1.5 that for $x, y \in G$, the Green kernel $G(x, y)$ is the expected number of visits to y made by the random walk that starts at x. We will suppose that the random walks are irreducible and transient, so that $0 < G(x, y) < \infty$ for every $x, y \in G$. The *Martin kernel* of the μ-random walk on G is defined as

$$K(x, y) := \frac{G(x, y)}{G(e_G, y)}, \text{ for } x, y \in G.$$

Definition 3.2.20. The *Martin compactification* of G is the smallest metrizable compact space containing Γ as a discrete dense subset, and to which all functions $K(x,\cdot)$,$x \in G$, extend continuously. The *Martin boundary* $\mathcal{M}(G,\mu)$ is the complement of G in its Martin compactification.

Denote by $\mathcal{H}^+$ the space of μ-harmonic functions on H that only take positive values. A function $h \in \mathcal{H}^+$ is called *minimal* if $h(e_G) = 1$ and for every $\tilde{h} \in \mathcal{H}^+$ that satisfies $h \geq \tilde{h}$ pointwise, it holds that $\tilde{h}/h$ is constant. Minimal positive μ-harmonic functions are the extreme points of the convex set $\{h \in \mathcal{H}^+ \mid h(e_G) = 1\}$. It is known that every minimal positive μ-harmonic function on Γ is of the form $K(\cdot,\xi)$, for some $\xi \in \mathcal{M}(G,\mu)$. The set

$$\mathcal{M}_{\min}(G,\mu) \coloneqq \{\xi \in \mathcal{M}(G,\mu) \mid K(\cdot,\xi) \text{ is a minimal positive } \mu\text{-harmonic function}\}$$

is a Borel subset of $\mathcal{M}(G,\mu)$.

Every positive μ-harmonic function has a unique integral representation with respect to a Borel measure on the set of minimal positive μ-harmonic functions. More precisely, we have the following.

Theorem 3.2.21 ([Sawyer, 1997, Theorem 4.1])**.** *Let G be a finitely generated group and let μ be a probability measure on G that induces a transient random walk. Then for every $h \in \mathcal{H}^+$ there exists a unique Borel measure ν^h on $\mathcal{M}(G,\mu)$ whose support is contained in $\mathcal{M}_{\min}(G,\mu)$ and satisfies*

$$h(g) = \int_{\mathcal{M}(G,\mu)} K(g,\xi)\,\mathrm{d}\nu^h(\xi), \text{ for } g \in G.$$

The Martin boundary can be endowed with an appropriate harmonic measure in order to provide a model of the Poisson boundary of (G,μ).

Theorem 3.2.22. *Let ν_1 be the measure on $\mathcal{M}(G,\mu)$ from Theorem 3.2.21 associated with the constant function $\mathbb{1} \in \mathcal{H}_+$, where $\mathbb{1}(g) = 1$ for every $g \in G$. Then $(\mathcal{M}(G,\mu),\nu_1)$ is a model for the Poisson boundary of (G,μ).*

The proof of this result can be found in [Woess, 2000, Theorem 24.12]; see also [Kaimanovich, 1996, Section 2.2] and [Sawyer, 1997, Section 1].

The probability measure ν_1 on the Martin boundary $\mathcal{M}(G,\mu)$ is not in general fully supported. For example, if μ is a non-degenerate finitely supported probability measure on $\mathbb{Z}^d$ with non-zero drift, then the associated Martin boundary is homeomorphic to the unit sphere S^{d-1} [NeySpitzer, 1966, Corollary 1.3] (see also [Woess, 2000, Corollary 25.16]). In contrast, Theorem 3.2.9 guarantees that the associated Poisson boundary is trivial. Another example in the context of non-trivial boundaries can be found in the family of free groups. On the one hand, it is proved in [Kaimanovich, 2000, Theorem 7.7] that the Poisson boundary of any non-degenerate probability measure with finite entropy and finite first logarithmic moment on a non-elementary free group is isomorphic to the space of infinite reduced words endowed with the corresponding harmonic measure. On the other hand, there exist measures as above on free groups such that the corresponding Martin boundary is homeomorphic to the circle S^1 [BallmannLedrappier, 1996]. Furthermore, it is known that on any finitely generated non-amenable group there are uncountably many possible different Martin boundaries for measures with exponential tails [Gouëzel, 2015, Theorem 1.1]. In particular, this result implies that on any non-elementary hyperbolic group there are non-degenerate measures with a finite exponential moment and for which the Martin boundary is not homeomorphic to the Gromov boundary of the group.

3.3 Amenability and Choquet-Deny groups

The existence of non-degenerate probability measures on a group G that have a (non-)trivial Poisson boundary is connected to the (non-)amenability of the group (see Section 1.3 for the definition and the properties of amenability that are relevant to this thesis).

Theorem 3.3.1 ([Azencott, 1970; Furstenberg, 1973; KaimanovichVershik, 1983; Rosenblatt, 1981])**.** *A group G is amenable if and only if it admits a non-degenerate random walk with a trivial Poisson boundary.*

We already saw in Corollary 3.2.14 that every non-degenerate probability measure μ on a non-amenable group G admits non-constant bounded μ-harmonic functions. This is equivalent to saying that the Poisson boundary of (G, μ) is non-trivial. This result was first proved in [Azencott, 1970, Corollaire after Proposition II.1] (see also the last two paragraphs of Section 9 in [Furstenberg, 1973]). The converse of this result is due to [Rosenblatt, 1981, Theorem 1.10] and [KaimanovichVershik, 1983, Theorem 4.4] who showed that if G is amenable, then one can construct a probability measure μ on G with $\mathrm{supp}(\mu) = G$ and a trivial Poisson boundary by considering an infinite convex combination of uniform measures on sufficiently fast-growing Følner sets.

3.3.1 Choquet-Deny groups

Theorem 3.3.1 states that the amenability of a group is completely determined by the existence of *some* non-degenerate random walk with a trivial Poisson boundary. A special case of this are groups for which *every* non-degenerate random walk has this property.

Definition 3.3.2. A countable group G is said to be *Choquet-Deny* if every non-degenerate measure has the Liouville property.

Every abelian group is Choquet-Deny [Blackwell, 1955; ChoquetDeny, 1960; DoobSnell-Williamson, 1960] (see also Theorem 3.2.9), and more generally so is any nilpotent group [DynkinMaljutov, 1961; Margulis, 1966] and every hyper-FC-central group [Jaworski, 2004; LinZaidenberg, 1998]. Recall that a group is said to be ICC if every non-trivial element in the associated quotient group has an infinite conjugacy class. The hyper-FC-center of a group G is the minimal normal subgroup of G with the property that the associated quotient group is ICC. A group G is said to be hyper-FC-central if it coincides with its hyper-FC-center. In particular, a finitely generated group is hyper-FC-central if and only if it is virtually nilpotent [McLain, 1956, Theorem 2], [DuguidMcLain, 1956, Theorem 2] (see also [FrischFerdowsi, 2018] for a self-contained presentation of the proof of this result).

Recently, it was proved by Frisch-Hartman-Tamuz-Vahidi Ferdowski that any countable group that is not hyper-FC-central admits a non-degenerate probability measure with a non-trivial Poisson boundary [FrischHartmanTamuzVahidi Ferdowsi, 2019]. Hence, the class of countable Choquet-Deny groups is precisely the one of countable hyper-FC-central groups.

This result has been further developed in [ErschlerKaimanovich, 2023], where it is shown that any ICC group G satisfies the following property: There exists a non-degenerate symmetric probability measure μ on G with finite entropy, whose Poisson boundary can be completely described in terms of the convergence of sample paths to the boundary of a locally finite forest, whose vertex set is G.

3.4 Entropy

Definition 3.4.1. The Shannon *entropy* of a probability measure μ on G is defined as

$$H(\mu) \coloneqq -\sum_{g\in G} \mu(g)\log\mu(g),$$

with the convention $0 \cdot \log(0) = 0$. We say that μ has finite entropy if $H(\mu) < \infty$.

The following is a fundamental inequality for the entropy of a finitely supported probability measure.

Lemma 3.4.2. *Let G be a group and μ a probability measure on G. Suppose that $|\text{supp}(\mu)| = N < \infty$. Then $H(\mu) \leq \log(N)$, and equality holds if and only $\mu(g) = 1/N$ for every $g \in \text{supp}(\mu)$.*

The fact that the function $\kappa : [0,1] \to \mathbb{R}$ defined by $\kappa(x) \coloneqq -x\log(x)$ is strictly convex implies the following lemma.

Lemma 3.4.3. *Let μ_1, μ_2 be probability measures on G. Then*

$$H(\mu_1 * \mu_2) \leq H(\mu_1) + H(\mu_2).$$

In particular, it is a consequence of Lemma 3.4.3 that the sequence $\{H(\mu^{*n})\}_{n\geq 1}$ of entropies of the n-fold convolutions of μ is subadditive. Thus, $\lim_{n\to\infty} \frac{H(\mu^{*n})}{n}$ exists and is equal to $\inf_{n\geq 1} \frac{H(\mu^{*n})}{n}$.

Definition 3.4.4 ([Avez, 1972])**.** The value

$$h(G,\mu) \coloneqq \lim_{n\to\infty} \frac{H(\mu^{*n})}{n}$$

is called the *asymptotic* or *Avez entropy* of the random walk (G, μ).

The value of $h(G, \mu)$ can be interpreted as the asymptotic mean quantity of information that is contained in one factor of the product $g_1 g_2 \cdots g_n$ of n independent random variables $g_1, \ldots, g_n$ distributed according to μ.

The asymptotic entropy can also be obtained by taking limits through a subset of full measure of the space of trajectories, thanks to the following Shannon-McMillan-Breiman result.

Theorem 3.4.5 ([Derriennic, 1980, Section 4], [KaimanovichVershik, 1983, Theorem 2.1])**.** *Suppose that $H(\mu) < \infty$. Then for $\mathbb{P}$-a.e. $\mathbf{w} = (w_0, w_1, \ldots) \in G^{\mathbb{Z}_+}$,*

$$h(G, \mu) = \lim_{n\to\infty} -\frac{1}{n} \log p_n(w_n).$$

This convergence also holds in $L^1(G^{\mathbb{Z}_+}, \mathbb{P})$.

Proof. This result follows from Kingman's Subadditive Ergodic Theorem [Kingman, 1968]. Using the same notation as in Theorem 3.1.2, let us define the non-negative random variables $a_n(\mathbf{g}) \coloneqq -\log(p_n(g_1 g_2 \cdots g_n))$, for $\mathbf{g} \in G^{\mathbb{Z}_+}$. Then we have

$$\mathbb{E}[a_1] = \sum_{g\in G} -\log(\mu(g))\mu(g) = H(\mu) < \infty.$$

In addition, for any $n, m \geq 1$, we have

$$p_{n+m}(g_1 g_2 \cdots g_n g_{n+1} \cdots g_{n+m}) \geq p_n(g_1 g_2 \cdots g_n) p_m(g_{n+1} \cdots g_{n+m}).$$

This implies that

$$\begin{aligned} a_{n+m}(\mathbf{g}) &= -\log(p_{n+m}(w_{n+m})) \\ &\leq -\log(p_n(w_n)) - \log(p_m(g_{n+1} \cdots g_{n+m})) \\ &= a_n(\mathbf{g}) + a_m(\sigma^n \mathbf{g}). \end{aligned}$$

Theorem 3.1.2 implies that there exists a constant A such that for $\mathbb{P}$-almost every trajectory $\mathbf{w} \in G^{\mathbb{Z}_+}$, we have

$$A = \lim_{n\to\infty} -\frac{1}{n} \log p_n(w_n),$$

and this convergence also holds in $L^1(G^{\mathbb{Z}_+}, \mathbb{P})$. In particular, it holds that

$$A = \lim_{n\to\infty} -\frac{1}{n} \mathbb{E}\left[\log p_n(w_n)\right] = \lim_{n\to\infty} -\frac{1}{n} H(\mu^{*n}) = h(G, \mu).$$

□

Remark 3.4.6. It follows from Theorem 3.4.5 that for every $\varepsilon > 0$, the following holds:

1. there exists a sequence of finite subsets $A_n \subseteq G$ with $|A_n| \leq \exp\left((h(G, \mu) + \varepsilon)n\right)$, such that $\mathbb{P}$-almost surely $w_n \in A_n$ for n sufficiently large, and
2. for any sequence of finite subsets $B_n \subseteq G$ with $|B_n| \leq \exp\left((h(G, \mu) - \varepsilon)n\right)$, it holds that $\mathbb{P}$-almost surely $w_n \notin B_n$ for n sufficiently large.

3.4.1 The entropy criterion

Avez proved in that any random walk on a group G with a finitely supported step distribution μ such that $h(G, \mu) = 0$ has a trivial Poisson boundary [Avez, 1972]. More generally, the following criterion, due to Derriennic and Kaimanovich-Vershik, completely characterizes probability measures with finite entropy and with $h(G, \mu) = 0$ as those that have a trivial Poisson boundary.

Theorem 3.4.7 ([Derriennic, 1980], [KaimanovichVershik, 1983, Theorem 1.1])**.** *Let μ be a probability measure on a group G such that $H(\mu) < \infty$. Then $h(G, \mu) = 0$ if and only if (G, μ) has a trivial Poisson boundary.*

A way of applying the entropy criterion is to show that the trajectories of the random walk are confined to finite subsets of the group which grow subexponentially, with positive probability bounded away from 0. This is made precise in the following corollary.

Corollary 3.4.8. *Let G be a group and let μ a probability measure on G such that $H(\mu) < \infty$. Suppose that there exist $\varepsilon > 0$ and finite subsets $A_n \subseteq G$, $n \geq 1$, such that $p_n(A_n) > \varepsilon$ for all $n \geq 1$ and $\lim_{n\to\infty} \frac{1}{n} \log |A_n| = 0$. Then the Poisson boundary of (G, μ) is trivial.*

Proof. Looking for a contradiction, let us suppose that $h := h(G, \mu) > 0$. For each $n \geq 1$, define the set

$$E_n := \left\{ \mathbf{w} \in G^{\mathbb{Z}_+} \mid -\frac{1}{n} \log p_n(w_n) > h/2 \right\}.$$

Thanks to Theorem 3.4.5, we know that for $\mathbb{P}$-almost every trajectory $\mathbf{w} \in G^{\mathbb{Z}_+}$ it holds that $-\frac{1}{n} \log p_n(w_n) \xrightarrow[n\to\infty]{} h$. This implies that $\mathbb{P}(E_n) \xrightarrow[n\to\infty]{} 0$.

Let us also note that

$$p_n(A_n \cap C^n(E_n^c)) = \sum_{a \in A_n \cap C^n(E_n^c)} p_n(a) \leq |A_n| \exp(-nh/2) \xrightarrow[n\to\infty]{} 0.$$

However, this is a contradiction since $\liminf_{n\to\infty} p_n(A_n \cap C^n(E_n^c)) \geq \varepsilon$. □

Remark 3.4.9. The entropy criterion states that the information of the (non-)triviality of the Poisson boundary of the random walk is entirely contained in one single numerical value, defined in terms of the convolution powers of μ. This is not necessarily the case for measures with $H(\mu) = \infty$, as we explain now.

For a probability measure μ on G, define the *reflected probability measure* $\check{\mu}$ on G by $\check{\mu}(g) = \mu(g^{-1})$, for $g \in G$. If one forgets about the group structure of G, the values of the convolution powers of μ and $\check{\mu}$ coincide (as measures on a countable set). However, it is possible to construct examples of probability measures μ on a group G with $H(\mu) = \infty$, such that (G, μ) has a trivial Poisson boundary and $(G, \check{\mu})$ has a non-trivial Poisson boundary [KaimanovichVershik, 1983, Example 6.5]. The entropy criterion guarantees that for probability measures of finite entropy, the μ-random walk and the $\check{\mu}$-random walk on G have either trivial or non-trivial Poisson boundaries simultaneously.

3.4.2 The relation between entropy, speed and volume growth

Suppose that G is a finitely generated group, and let S be a finite generating set of G. Consider the corresponding word length $|\cdot|_S$ on G.

Definition 3.4.10. A probability measure μ on G is said to have a *finite α-moment*, where $\alpha > 0$, if $\sum_{g\in G} |g|_S^\alpha \mu(g) < \infty$.

It is well-known that probability measures on a finitely generated group with a finite first moment must have finite entropy. We state this fact and provide its proof below.

Lemma 3.4.11 ([Derriennic, 1986], [Kaimanovich, 2001, Lemma 2.2.2])**.** *Let G be a finitely generated group and μ a probability measure on G with a finite first moment. Then $H(\mu) < \infty$.*

Proof. Let us fix a finite generating set S of G, and denote $l : G \to \mathbb{Z}_+$ the word length map $l(g) := |g|_S$, for $g \in G$. Denote $\nu := l_*\mu$ the push-forward probability measure on $\mathbb{Z}_+$.

For every $r \geq 0$ let us denote $S_r := \{g \in G \mid l(g) = r\}$, and define $\alpha_r := \frac{1}{\mu(S_r)}\mu|_{S_r}$ the normalized restriction of μ to S_r. Denote that $\nu_r := \nu(\{r\}) = \mu(S_r)$.

With the above we have $\mu = \sum_{r\geq 0} \nu(\{r\})\alpha_r$, and hence

$$\begin{aligned} H(\mu) &= \sum_{g\in G} \kappa \left(\sum_{k\geq 0} \nu_r \alpha_r(g) \right) \\ &= \sum_{r\geq 0} \sum_{g\in S_r} \kappa\left(\nu_r \alpha_r(g)\right) \\ &= -\sum_{r\geq 0} \sum_{g\in S_r} \nu_r \alpha_r(g) \log\left(\nu_r \alpha_r(g)\right) \\ &= -\sum_{r\geq 0} \sum_{g\in S_r} \nu_r \alpha_r(g) \log\left(\nu_r\right) - \sum_{r\geq 0} \sum_{g\in S_r} \nu_r \alpha_r(g) \log\left(\alpha_r(g)\right) \\ &= H(\nu) + \sum_{r\geq 0} \nu_r H(\alpha_r). \end{aligned}$$

We will see that each of these two terms is finite. On the one hand, we have

$$\sum_{r\geq 0} \nu_r H(\alpha_r) \leq \sum_{r\geq 0} \nu_r \log|S_r| = \sum_{r\geq 0} \mu(S_r) \log|S_r| \leq \sum_{r\geq 0} r\mu(S_r) \log|S| = \log|S| \sum_{g\in G} \mu(g) l(g) < \infty.$$

On the other hand, using that $\kappa(x) = -x\log(x)$ is strictly increasing in $[0, 1/e]$, we have that

$$\begin{aligned} H(\nu) &= -\sum_{r\geq 0} \nu_r \log(\nu_r) \\ &\leq \sum_{r\geq 0} \nu_r \max\{-\log(\nu_r), r\} \\ &= \sum_{\substack{r\geq 0 \\ -\log(\nu_r)\leq r}} r\nu_r + \sum_{\substack{r\geq 0 \\ -\log(\nu_r) > r}} \kappa(\nu_r) \\ &\leq \sum_{r\geq 0} r\nu_r + \sum_{r\geq 0} \kappa(e^{-r}) \\ &= \sum_{g\in G} l(g)\mu(g) + \sum_{r\geq 0} re^{-r} < \infty. \end{aligned}$$

□

For measures with a finite first moment, it is possible to define the rate at which the word length of the random walk grows along sample paths.

Proposition 3.4.12. *Let G be a group and consider S a finite generating set of G. Let μ be a probability measure on G with a finite first moment, and consider $\{w_n\}_{n\geq 0}$ the μ-random walk on G. Then there exists $\ell \geq 0$ such that for $\mathbb{P}$-almost every $\mathbf{w} \in G^{\mathbb{Z}_+}$, we have*

$$\ell = \lim_{n\to\infty} \frac{|w_n|_S}{n}.$$

Additionally, it holds that $\ell := \lim_{n\to\infty} \frac{\mathbb{E}(|w_n|_S)}{n}$. The constant ℓ is called the speed *of the random walk (G, μ).*

Proof. This result is a consequence of Kingman's Subadditive Ergodic Theorem [Kingman, 1968]. Indeed, following the notation of Theorem 3.1.2, let us define the non-negative random variables $a_n(\mathbf{g}) := |g_1 g_2 \cdots g_n|_S$, for $\mathbf{g} \in G^{\mathbb{Z}_+}$. Since μ has a finite first moment, we have $\mathbb{E}[a_1] = \sum_{g\in G} |g|_S \mu(g) < \infty$. Additionally, using the triangular inequality of the word length we see that for any $n, m \geq 1$,

$$\begin{aligned} a_{n+m}(\mathbf{g}) &= |g_1 g_2 \cdots g_n g_{n+1} \cdots g_{n+m}|_S \\ &\leq |g_1 g_2 \cdots g_n|_S + |g_{n+1} \cdots g_{n+m}|_S \\ &= a_n(\mathbf{g}) + a_m(\sigma^n(\mathbf{g})). \end{aligned}$$

The hypotheses of Theorem 3.1.2 are hence verified and the conclusion of the proposition follow from it. □

Additionally, let us denote $v := \lim_{n\to\infty} \frac{1}{n} \log\Big(|\{g \in G \mid |g| \leq n\}| \Big)$ the *exponential growth rate* of G. Recall also that $h(G,\mu) = \lim_{n\to\infty} \frac{H(\mu^{*n})}{n}$ denotes the asymptotic entropy of (G, μ). It turns out that the asymptotic entropy $h(G, \mu)$, the speed ℓ and the exponential growth rate v are related by the following inequality.

Lemma 3.4.13 ([Guivarch, 1980, Proposition 2])**.** *Let G be a finitely generated group and μ a probability measure on G with a finite first moment. With the notation above, we have*

$$h(G, \mu) \leq \ell v.$$

This result was originally proved by Guivarc'h in [Guivarch, 1980] and is commonly referred to as the "fundamental inequality" (e.g. in [Vershik, 1999]).

The following result goes back to [Avez, 1976], who proved that every bounded harmonic function with respect to a finitely supported probability measure on a group of subexponential growth must be constant.

Proposition 3.4.14. *Let G be a group of subexponential growth and μ a probability measure on G with a finite first moment. Then (G, μ) has a trivial Poisson boundary.*

Proof. It follows from Lemma 3.4.11 that $H(\mu) < \infty$. Thanks to the entropy criterion (Theorem 3.4.7), in order to prove that (G, μ) has a trivial Poisson boundary it suffices to show that $h(G, \mu) = 0$. □

Due to Proposition 3.4.14, any probability measure on a group of subexponential growth that has a non-trivial Poisson boundary must necessarily have an infinite first moment. The first results on the existence of non-trivial Poisson boundaries on groups of intermediate growth are given in [Erschler, 2004a], and it follows from [FrischHartmanTamuzVahidi Ferdowsi, 2019] that such probability measures exist in every group of intermediate growth. Probability measures with non-trivial Poisson boundary on Grigorchuk's group are studied in [ErschlerZheng, 2020], where the construction of such measures with a control over their tail decay allows them to obtain near-optimal lower bounds for the asymptotic volume growth of Grigorchuk's group.

3.4.3 Entropy of partitions and the conditional entropy criterion

There is a conditional version of Theorem 3.4.7 proved by Kaimanovich, that allows to determine whether a given μ-boundary is the complete Poisson boundary. During the last decades, this criterion has been the main tool in the identification of non-trivial Poisson boundaries of groups. Below we state this criterion in two different formulations, and the associated geometric criteria that are deduced from it.

Entropy of partitions

We first introduce our notation for the partitions of the space of sample paths and their entropy.

Let $\rho = \{\rho_k\}_{k\geq 1}$ be a countable partition of the space of sample paths $G^{\mathbb{Z}_+}$. The entropy of ρ with respect to the measure $\mathbb{P}$ is defined as

$$H(\rho) \coloneqq -\sum_{k\geq 1} \mathbb{P}(\rho_k) \log \mathbb{P}(\rho_k).$$

The following sequence of partitions will be the most important for the statements below.

Definition 3.4.15. For every $n \geq 1$, define the partition α_n of the space of sample paths $G^{\mathbb{Z}_+}$, where two trajectories $\mathbf{w}, \mathbf{w}' \in G^{\mathbb{Z}_+}$ belong to the same element of α_n if and only if $w_n = w_n'$.

That is, α_n is the partition given by the n-th instant of the μ-random walk. Note that we have

$$H(\alpha_n) = -\sum_{g\in G} \mathbb{P}(w_n = g) \log \mathbb{P}(w_n = g) = H(\mu^{*n}),$$

so that the entropy of the partition α_n coincides with the entropy of the convolution power μ^{*n}, which is the law of the n-th step of the μ-random walk on G. From this, the asymptotic entropy h_μ of the random walk (G, μ) can be expressed in terms of the partitions α_n, $n \geq 1$, as $h(G, \mu) = \lim_{n\to\infty} \frac{H(\alpha_n)}{n}$.

Let $\mathbf{X} = (X, \lambda)$ be a μ-boundary with boundary map $\mathrm{bnd}_{\mathbf{X}} : (G^{\mathbb{Z}_+}, \mathbb{P}) \to (X, \lambda)$. Then the measure $\mathbb{P}$ can be disintegrated with respect to the map $(G^{\mathbb{Z}_+}, \mathbb{P}) \to (X, \lambda)$. That is, for λ-a.e. $\xi \in X$ there exists the *conditional probability measure* $\mathbb{P}^\xi$ on $G^{\mathbb{Z}_+}$, supported on $\mathrm{bnd}_{\mathbf{X}}^{-1}(\{\xi\})$, and it holds that

$$\mathbb{P} = \int_X \mathbb{P}^\xi \,\mathrm{d}\lambda(\xi).$$

We define the conditional entropy of a partition in an analogous manner to Definition 3.4.24.

Definition 3.4.16. Consider $\rho = \{\rho_k\}_{k\geq 1}$ a countable partition of the space of sample paths $G^{\mathbb{N}}$ and let $\mathbf{X} = (X, \lambda)$ be a μ-boundary of G. For λ-a.e. $\xi \in X$, we define the *conditional entropy of ρ given ξ* as

$$H_\xi(\rho) \coloneqq -\sum_{k\geq 1} \mathbb{P}^\xi(\rho_k) \log \mathbb{P}^\xi(\rho_k).$$

Following [Rohlin, 1967, Section 5.1], let us define the *mean conditional entropy over the μ-boundary* $\mathbf{X}$ as

$$H_{\mathbf{X}}(\rho) \coloneqq \int_X H_\xi(\rho) \,\mathrm{d}\lambda(\xi).$$

An important fact about the mean conditional entropy of a partition over a μ-boundary is at most the entropy of the partition. We state this in the following lemma.

Lemma 3.4.17. *Consider a countable partition ρ of the space of sample paths $G^{\mathbb{N}}$. Let $\mathbf{X}$ be a μ-boundary of G and $\mathbf{Y}$ a quotient of $\mathbf{X}$ with respect to a G-equivariant measurable partition. Then it holds that $H_{\mathbf{X}}(\rho) \leq H_{\mathbf{Y}}(\rho)$.*

In particular for every countable partition ρ and each μ-boundary $\mathbf{X}$, we have that $H_{\mathbf{X}}(\rho) \leq H(\rho)$.

This follows from the fact that the function $x \mapsto -x\log(x)$ is concave on $[0, 1]$, together with Jensen's inequality and the fact that conditional entropies are non-negative (see [Rohlin, 1967, Section 5.10]).

Consider two countable partitions ρ, γ of the path space $G^{\mathbb{Z}_+}$. Let (X, λ) be a μ-boundary of G. For every $\xi \in X$, denote

$$\mathbb{P}^\xi(P \mid Q) \coloneqq \frac{\mathbb{P}^\xi(P \cap Q)}{P^\xi(Q)}, \text{ for } P \in \rho \text{ and } Q \in \gamma \text{ with } P^\xi(Q) \neq 0.$$

We define the associated *conditional entropy of ρ with respect to γ conditioned on ξ* as

$$H_\xi(\rho \mid \gamma) \coloneqq -\sum_{P\in\rho, Q\in\gamma} \mathbb{P}^\xi(P \cap Q) \log \mathbb{P}^\xi(P \mid Q).$$

Define also

$$H_{\mathbf{X}}(\rho \mid \gamma) \coloneqq \int_X H_\xi(\rho \mid \gamma) \,\mathrm{d}\lambda(\xi).$$

We have the equality $H_{\mathbf{X}}(\rho \mid \gamma) = H_{\mathbf{X}}(\rho \vee \gamma) - H_{\mathbf{X}}(\gamma)$, where $\rho \vee \gamma \coloneqq \{P \cap Q \mid P \in \rho, Q \in \gamma\}$ is the *join* of the partitions ρ and γ.

Remark 3.4.18. Rokhlin's theory of measurable partitions [Rohlin, 1967] implies that for every μ-boundary $\mathbf{X}$ of G, there is an associated measurable partition η of the space of sample paths $G^{\mathbb{Z}_+}$. With this, one can express the mean conditional entropy of a countable partition ρ over the μ-boundary $\mathbf{X}$ as $H_{\mathbf{X}}(\rho) = H(\rho \mid \eta)$. Similarly, for two countable partitions ρ, γ of the space of sample paths, we have $H_{\mathbf{X}}(\rho \mid \gamma) = H(\rho \mid \gamma \vee \eta)$. This is the notation used in [Kaimanovich, 1985; KaimanovichVershik, 1983] when the mean conditional entropy is discussed.

In this thesis (particularly in Chapter 5) we use the notation $H_{\mathbf{X}}(\rho \mid \gamma)$ for the mean conditional entropy, in order to emphasize the dependence on the μ-boundary that is being considered.

Throughout this thesis, we will use basic facts about the (conditional) entropy of partitions. In particular, we will use the following inequalities in Section 3.6 when we explain the approach to the conditional entropy criterion via the "pin-down approximation". This has been used in the articles [ChawlaForghaniFrischTiozzo, 2022; ForghaniTiozzo, 2019], and it will be used in this thesis in Chapter 5.

Lemma 3.4.19. *Consider countable partitions ρ, γ and δ of $G^{\mathbb{Z}_+}$, and a μ-boundary $\mathbf{X}$. The following properties hold.*

1. $H_{\mathbf{X}}(\rho \mid \delta) = H_{\mathbf{X}}(\rho \mid \gamma \vee \delta) + H_{\mathbf{X}}(\gamma \mid \delta)$.
2. $H_{\mathbf{X}}(\rho \mid \gamma) \leq H_{\mathbf{X}}(\rho \vee \delta \mid \gamma)$.
3. $H_{\mathbf{X}}(\rho \mid \gamma \vee \delta) \leq H_{\mathbf{X}}(\rho \mid \gamma)$.

We refer to [MartinEngland, 1981, Corollaries 2.5 and 2.6] and [Rohlin, 1967, Section 5] for the proofs of these properties.

Kaimanovich's conditional entropy criterion

Let $\mathbf{X} = (X, \lambda)$ be a μ-boundary of G with boundary map $\mathrm{bnd}_{\mathbf{X}} : (G^{\mathbb{Z}_+}, \mathbb{P}) \to (X, \lambda)$, and recall that for λ-a.e. $\xi \in X$ we denote by $\mathbb{P}^{\xi}$ the conditional probability measure on $G^{\mathbb{Z}_+}$, with respect to the measurable partition induced by the boundary map $\mathrm{bnd}_{\mathbf{X}}$. It can be shown that for λ-a.e. $\xi \in X$, the probability measure $\mathbb{P}^{\xi}$ is the law of a Markov process on G, with transition probabilities

$$p^{\xi}(h, g) = \mu(h^{-1}g)\frac{\mathrm{d}g_*\lambda}{\mathrm{d}h_*\lambda}(\xi) \tag{3.5}$$

for $g, h \in G$; see Equation 20 in page 462 of [KaimanovichVershik, 1983].

Definition 3.4.20. Let us define the *differential entropy* of the μ-boundary $\mathbf{X} = (X, \lambda)$ by

$$E(\mathbf{X}) := \int_{G^{\mathbb{Z}_+}} \log\left(\frac{\mathrm{d}w_{1*}\lambda}{\mathrm{d}\lambda}(\mathrm{bnd}_{\mathbf{X}}(\mathbf{w}))\right) \mathrm{d}\mathbb{P}(\mathbf{w}).$$

The differential entropy of a μ-boundary can be expressed in terms of the entropy of μ, and the mean conditional entropy. The following proposition goes back to [Kaimanovich, 1985], and it is a particular case of [Kaimanovich, 2000, Lemma 4.2]. Recall that α_1 denotes the partition of the space of sample paths $G^{\mathbb{Z}_+}$ determined by the first step of the μ-random walk (Definition 3.4.15).

Proposition 3.4.21. *Let μ be a probability measure on G such that $H(\mu) < \infty$, and let $\mathbf{X}$ be a μ-boundary of G. Then $E(\mathbf{X}) = H(\mu) - H_{\mathbf{X}}(\alpha_1)$.*

Proof. Using Equation (3.5), we have that

$$\begin{aligned}
E(\mathbf{X}) &= \int_{G^{\mathbb{Z}_+}} \log\left(\frac{\mathrm{d}w_{1*}\lambda}{\mathrm{d}\lambda}(\mathrm{bnd}_{\mathbf{X}}(\mathbf{w}))\right) \mathrm{d}\mathbb{P}(\mathbf{w}) \\
&= \int_{G^{\mathbb{Z}_+}} \log\left(\frac{p^{\mathrm{bnd}_{\mathbf{X}}(\mathbf{w})}(e_G, w_1)}{\mu(w_1)}\right) \mathrm{d}\mathbb{P}(\mathbf{w}) \\
&= -\int_{G^{\mathbb{Z}_+}} \log(\mu(w_1))\, \mathrm{d}\mathbb{P}(\mathbf{w}) + \int_{G^{\mathbb{Z}_+}} \log\left(p^{\mathrm{bnd}_{\mathbf{X}}(\mathbf{w})}(e_G, w_1)\right) \mathrm{d}\mathbb{P}(\mathbf{w}) \\
&= H(\mu) + \int_X \int_{G^{\mathbb{Z}_+}} \log\left(p^{\xi}(e_G, w_1)\right) \mathrm{d}\mathbb{P}^{\xi}(\mathbf{w})\, \mathrm{d}\lambda(\xi) \\
&= H(\mu) - \int_X H_{\xi}(\alpha_1)\, \mathrm{d}\lambda(\xi) \\
&= H(\mu) - H_{\mathbf{X}}(\alpha_1).
\end{aligned}$$

□

Recall that we denote by $(\partial_\mu G, \nu)$ the Poisson boundary of the μ-random walk on G. The following result characterizes the Poisson boundary by the property of maximizing the value of the differential entropy. This result is proved in [KaimanovichVershik, 1983], and the argument can be traced back to [Furstenberg, 1971, Lemma 5.6], where Furstenberg studied the Poisson boundary of random walks on $\mathrm{SL}(n, \mathbb{Z})$.

Theorem 3.4.22 ([KaimanovichVershik, 1983, Theorem 3.2])**.** *Let μ be a probability measure on a countable group G, with $H(\mu) < \infty$. Then for every μ-boundary $\mathbf{X}$, it holds that $E(\mathbf{X}) \leq h(G, \mu)$. Furthermore, $E(\mathbf{X}) = h(G, \mu)$ if and only if $\mathbf{X} \cong (\partial_\mu G, \nu)$.*

Combining this result with Proposition 3.4.21 and the fact that $H_{\partial_\mu G}(\alpha_1) = H(\mu) - h(G, \mu)$ (see Equation 13 in page 465 in [KaimanovichVershik, 1983]), we obtain the following.

Corollary 3.4.23. *Let μ be a probability measure on a countable group G, with $H(\mu) < \infty$. A μ-boundary $\mathbf{X}$ is the Poisson boundary if and only if $H_{\mathbf{X}}(\alpha_1) = H_{\partial_\mu G}(\alpha_1)$.*

See also [Kaimanovich, 2001, Theorem 4.3], where it is shown that the mean conditional entropy is monotonous, in the following sense: if $\mathbf{X}_1$ is a G-equivariant quotient of the Poisson boundary of (G, μ), and $\mathbf{X}_2$ is a G-equivariant quotient of $\mathbf{X}_1$ (so that both $\mathbf{X}_1$ and $\mathbf{X}_2$ are μ-boundaries), then $H_{\mathbf{X}_2}(\alpha_1) \leq H_{\mathbf{X}_1}(\alpha_1)$. Furthermore, equality holds if and only if $\mathbf{X}_1 \cong \mathbf{X}_2$.

Definition 3.4.24. Let G be a group and let μ be a probability measure on G, and consider $\mathbf{X} = (X, \lambda)$ a μ-boundary of G. Let us denote for λ-a.e. $\xi \in X$ the one-dimensional distributions of $\mathbb{P}^{\xi}$ as $p_n^{\xi}(g) := \mathbb{P}^{\xi}(w_n = g)$, for every $n \geq 1$ and $g \in G$. We define the *conditional asymptotic entropies* of (G, μ) by

$$h^{\xi}(G, \mu) := \lim_{n \to \infty} \frac{H\left(p_n^{\xi}\right)}{n}$$

for λ-almost every $\xi \in X$.

The following result relates the conditional asymptotic entropies of the μ-random walk to the mean conditional entropy.

Theorem 3.4.25 ([Kaimanovich, 1985, Theorem 1], [Kaimanovich, 2001, Theorem 4.5])**.** *Let G be a countable group and μ a probability measure with $H(\mu) < \infty$. Let $\mathbf{X} = (X, \lambda)$ be a μ-boundary of G. Then for $\mathbb{P}^{\xi}$-a.e. $\mathbf{w} = (w_0, w_1, \dots) \in G^{\mathbb{Z}_+}$,*

$$h^{\xi}(G, \mu) = \lim_{n\to\infty} -\frac{1}{n}\log(p_n^{\xi}(w_n)).$$

This convergence also holds in $L^1(G^{\mathbb{Z}_+}, \mathbb{P}^{\xi})$. Furthermore, one has that

$$h^{\xi}(G, \mu) = H_{\mathbf{X}}(\alpha_1) - H_{\partial_\mu G}(\alpha_1).$$

By combining Theorem 3.4.25 with Corollary 3.4.23, one obtains Kaimanovich's conditional entropy criterion.

Theorem 3.4.26 ([Kaimanovich, 1985, Theorem 2], [Kaimanovich, 2000, Theorem 4.6])**.** *Consider G a countable group, and let μ be a probability measure on G with $H(\mu) < \infty$. Consider a μ-boundary $\mathbf{X} = (X, \lambda)$. Then $\mathbf{X}$ is the Poisson boundary of (G, μ) if and only if $h^{\xi}(G, \mu) = 0$ for λ-almost every $\xi \in X$.*

Another way of stating Kaimanovich's conditional entropy criterion is the following, which uses the mean conditional entropy. Recall from Definition 3.4.15 that α_n is the partition of the space of sample paths $G^{\mathbb{Z}_+}$ determined by the position at instant n.

Theorem 3.4.27. *Let G be a countable group, and let μ be a probability measure on G with $H(\mu) < \infty$. Consider a μ-boundary $\mathbf{X} = (X, \lambda)$ of G. Then $\mathbf{X}$ is the Poisson boundary of (G, μ) if and only if*

$$\lim_{n\to\infty} \frac{H_{\mathbf{X}}(\alpha_n)}{n} = 0.$$

This formulation is already implicitly contained in [Kaimanovich, 1985] and, similarly to the conditional entropy criterion stated in Theorem 3.4.26, it is proved using Theorem 3.4.22 above. Indeed, from Equation (3.5) for the transition probabilities of the conditional random walks, we see that

$$\begin{aligned} H_{\mathbf{X}}(\alpha_n) &= -\int_{G^{\mathbb{Z}_+}} \log\left(\mu^{*n}(w_n)\frac{\mathrm{d}w_{n*}\lambda}{\mathrm{d}\lambda}(\mathrm{bnd}_{\mathbf{X}}(\mathbf{w}))\right)\mathrm{d}\mathbb{P}(\mathbf{w}) \\ &= -\int_{G^{\mathbb{Z}_+}} \log\left(\mu^{*n}(w_n)\right)\mathrm{d}\mathbb{P}(\mathbf{w}) + \int_{G^{\mathbb{Z}_+}} \log\left(\frac{\mathrm{d}w_{n*}\lambda}{\mathrm{d}\lambda}(\mathrm{bnd}_{\mathbf{X}}(\mathbf{w}))\right)\mathrm{d}\mathbb{P}(\mathbf{w}). \end{aligned}$$

The first term equals $H(\mu^{*n})$, whereas the second term equals $-nE(\mathbf{X})$ (see the proof of [Kaimanovich, 2000, Lemma 4.2]). Thus, we get that

$$\frac{1}{n}H_{\mathbf{X}}(\alpha_n) = \frac{1}{n}H(\mu^{*n}) - E(\mathbf{X}) \xrightarrow[n\to\infty]{} h(G, \mu) - E(\mathbf{X}).$$

From this equality, one sees that Theorem 3.4.27 is a consequence of Theorem 3.4.22.

We also mention that Theorem 3.4.27 is a particular case of [KaimanovichSobieczky, 2012, Theorem 2.17], where the result is proved in a more general setting for random walks along classes of graphed equivalence relations. For random walks on groups, the equivalence relation that one considers is the one induced by identifying sample paths of $G^{\mathbb{Z}_+}$ which are mapped to the same boundary point $\xi \in X$.

In Chapter 5 we will use this formulation of the conditional entropy criterion (Theorem 3.4.27), which has been also used in recent papers related to the identification of the Poisson boundary, such as [ChawlaForghaniFrischTiozzo, 2022; ForghaniTiozzo, 2019].

Geometric criteria deduced from the conditional entropy criterion

Given a group G, a probability measure μ on G with finite entropy, and a μ-boundary (X, λ), one of the main difficulties for verifying the hypothesis of the conditional entropy criterion is that, in general, one cannot obtain explicit expressions for the conditional entropies $\mathbb{P}^\xi$, for $\xi \in X$. As a consequence, it is not clear how to estimate appropriately the values of the conditional entropies $H_\xi(w_n)$, or the mean conditional entropy $H_{\mathbf{X}}(w_n)$, $n \geq 1$.

One way of applying this criterion is via the following corollary, whose proof is analogous to that of Corollary 3.4.8.

Corollary 3.4.28. *Let G be a group and let μ be a probability measure on G with $H(\mu) < \infty$. A μ-boundary $\mathbf{X} = (X, \lambda)$ is the Poisson boundary if for λ-a.e. $\xi \in X$, there exist $\varepsilon > 0$ and a sequence of finite sets $A_n \subseteq G$, $n \geq 1$, such that*

1. $\lim_{n\to\infty} \frac{1}{n} \log |A_n| = 0$, *and*
2. $\limsup_{n\to\infty} p_n^\xi(A_n) > \varepsilon$.

Lyons and Peres have proved an enhanced version of the conditional entropy criterion in [LyonsPeres, 2021a, Corollary 2.3], which states that one can replace the second item of Corollary 3.4.28 with the condition

$$\limsup_{n\to\infty} \mathbb{P}^\xi \left(\text{there is } m \geq n \text{ such that } w_m \in A_n\right) > \varepsilon.$$

We refer to [LyonsPeres, 2021a, Corollary 2.3] and [LyonsPeres, 2021b, Proposition 14.42] for its proof, which follows a similar structure to the proofs of Corollaries 3.4.8 and 3.4.28 via the Shannon-McMillan-Breiman theorem for conditional probabilities (Theorem 3.4.25). The enhanced criterion was used by Lyons and Peres in [LyonsPeres, 2021a, Theorem 3.1] to give a short proof of the description of the Poisson boundary for simple random walks on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$, $d \geq 3$. Furthermore, in [LyonsPeres, 2021a, Theorem 1.1] they used the enhanced criterion to provide the description of the Poisson boundary for random walks on wreath products $A \wr B$ for A and B any two countable groups, for adapted probability measures μ that have finite entropy and bounded lamp range (that is, at each step of the random walk the lamp configuration is modified only in a uniformly bounded neighborhood of the current position on B). These results are described in further detail in Subsection 3.5.1. This enhanced criterion has been further applied to the description of the Poisson boundary for other families of groups, such as the discrete affine group of a regular tree in [BrieusselTanakaZheng, 2021, Theorem 1.1] for random walks with a finitely supported step distribution.

Kaimanovich proved two geometric criteria in [Kaimanovich, 2000], known as the "ray criterion" and "strip criterion", which provide a concrete way of applying the conditional entropy criterion, with appropriate conditions on the step distribution μ. These results have been widely used in the past decades for the identification of the Poisson boundary, as we discuss in Section 3.5, and we recall their formulation below.

Theorem 3.4.29 (Ray Criterion, [Kaimanovich, 2000, Theorem 5.5])**.** *Let G be a finitely generated group, and let d denote the word metric with respect to some finite generating set of G. Let μ be a probability measure on G with $H(\mu) < \infty$. Consider (X, λ) a μ-boundary of G, with boundary map $\pi : (G^{\mathbb{Z}_+}, \mathbb{P}) \to (X, \lambda)$. Suppose that there exists a sequence of measurable maps $\varphi_n : (X, \lambda) \to G$ such that*

$$\frac{1}{n} d(w_n, \pi_n(\varphi_n(\mathbf{w}))) \xrightarrow[n\to\infty]{} 0$$

for $\mathbb{P}$-almost every trajectory $\mathbf{w} \in G^{\mathbb{Z}_+}$. Then (X, λ) is the Poisson boundary of (G, μ).

Given a probability measure μ on a group G, recall that we denote by $\check{\mu}$ the probability measure on G defined by $\check{\mu}(g) \coloneqq \mu(g^{-1})$, for $g \in G$,

Theorem 3.4.30 (Strip criterion, [Kaimanovich, 2000, Theorem 6.5])**.** *Let G be a finitely generated group, and let $|\cdot|$ denote the word length with respect to some finite generating set of G. Let μ be a probability measure on G. Consider (X, λ) a μ-boundary of G and $(\check{X}, \check{\lambda})$ a $\check{\mu}$-boundary of G. Suppose that there exists a measurable G-equivariant map S that assigns to each pair $(x, \check{x}) \in X \times \check{X}$ a non-empty subset $S(x, \check{x}) \subseteq G$. Suppose that either*

1. *the probability measure μ has a finite first moment and*

$$\lim_{n\to\infty} \frac{1}{n} \log |\{g \in S(x, \check{x}) \mid |g| \leq n\}| = 0,$$

for $\lambda \times \check{\lambda}$-almost every $(x, \check{x}) \in X \times \check{X}$, or

2. *the probability measure μ has a finite first logarithmic moment (i.e. $\sum_{g\in G} \log(|g|)\mu(g) < \infty$) and finite entropy, and*

$$\sup_{n>1} \frac{1}{\log(n)} \log |\{g \in S(x, \check{x}) \mid |g| \leq n\}| < \infty,$$

for $\lambda \times \check{\lambda}$-almost every $(x, \check{x}) \in X \times \check{X}$.

Then (X, λ) is the Poisson boundary of (G, μ) and $(\check{X}, \check{\lambda})$ is the Poisson boundary of $(G, \check{\mu})$.

3.5 Identification of Poisson boundaries

Given a group G and a probability measure μ on G, a natural question is to find an explicit model for the Poisson boundary of (G, μ). That is, to identify the Poisson boundary of the random walk with a concrete probability space endowed with a μ-stationary probability measure, on which G acts. Currently, the main tool to do this is Kaimanovich's conditional entropy criterion [Kaimanovich, 1985, Theorem 2], [Kaimanovich, 2000, Theorem 4.6] (Theorem 3.4.26), together with the ray criterion [Kaimanovich, 2000, Theorem 5.5] (Theorem 3.4.29) and strip criterion [Kaimanovich, 2000, Theorem 6.5] (Theorem 3.4.30) that are deduced from it. It is important to note that these criteria have two main restrictions: the first is that they rely on the measure μ having finite entropy, and hence they do not work in a general context for measures with infinite entropy. An exception to this can be found in [ForghaniTiozzo, 2019], where the Poisson boundary of free semigroups for some specific random walks with infinite entropy is described (although their methods are also based on entropy). This is explained in more detail in Subsection 3.6.2. The second restriction for the entropy criteria is that, in order to apply them, one needs to already know a μ-boundary of G that serves as a potential candidate for the Poisson boundary. There are families of groups for which one can prove that some random walks on them have a non-trivial boundary by using the entropy criterion (Theorem 3.4.7), but for which there is no known non-trivial μ-boundary.

We now make a summary of the known descriptions of (non-trivial) Poisson boundaries for various families of groups. We discuss the case of wreath products separately in Subsection 3.5.1, due to its importance for the results of this thesis. Additionally, many of the results discussed below are covered by the recent description of the Poisson boundary of acylindrically hyperbolic groups by [ChawlaForghaniFrischTiozzo, 2022, Theorem 1.2], which is discussed separately in Subsection 3.6.3.

The complete description of non-trivial Poisson boundaries goes back to [DynkinMaljutov, 1961], who proved that for any **non-abelian free group** and any probability measure supported on a free generating set, the corresponding Martin boundary can be identified with the space of infinite reduced words of the free group. This result was extended to all probability measures with finite support by [Derriennic, 1975]. More generally, [Ancona, 1987] proved that the Martin boundary of a **non-elementary hyperbolic group** with respect to a finitely supported measure coincides with its Gromov boundary. These results give a complete description of the Poisson boundary of free groups and hyperbolic groups for the corresponding probability measures, since the Martin boundary can always be endowed with a stationary probability measure so that it provides, as a probability space, a model for the Poisson boundary (see Subsection 3.2.5).

Kaimanovich developed a conditional entropy criterion (Theorem 3.4.26) which allowed him to describe the Poisson boundary of various families of groups, and for more general classes of measures than the ones that had been considered in the literature so far. He showed that the Poisson boundary of a **non-elementary hyperbolic group** for a non-degenerate measure with finite entropy and finite first logarithmic moment is fully described by the Gromov boundary endowed with the hitting measure [Kaimanovich, 1994, Theorem 8], [Kaimanovich, 2000, Theorems 7.4 and 7.7]. Recently, this description was proved to hold for all non-degenerate measures with finite entropy [ChawlaForghaniFrischTiozzo, 2022, Theorem 1.1]. This is the first time that a non-trivial Poisson boundary is identified for all measures with finite entropy, and due to the relevance of this result for this thesis, we discuss it in more detail in Subsection 3.6

below. We note that the two latter results were new even in the case of free groups. Additionally, [ChawlaForghaniFrischTiozzo, 2022, Theorem 1.2] identifies the Poisson boundary of **acylindrically hyperbolic groups** for probability measures with finite entropy, extending a result obtained previously on [MaherTiozzo, 2018, Theorem 1.5] with an extra assumption of a finite first logarithmic moment.

Kaimanovich also gave a description of the Poisson boundary for other classes of groups that can be endowed with natural geometric boundaries. This is the case for **groups with infinitely many ends**, where the associated geometric boundary is the space of ends. It is shown in [Kaimanovich, 2000, Theorem 8.4] that the space of ends endowed with an appropriate harmonic measures provides a model for the Poisson boundary of groups with infinitely many ends, for any non-elementary probability measure with finite entropy and finite first logarithmic moment. This result had been proved previously for finitely supported probability measures in [Woess, 1989, Theorem 7.1]. Kaimanovich also considered **co-compact lattices in rank one Cartan-Hadamard manifolds**, and proved that for non-degenerate probability measures with finite entropy and a finite first moment, the visual boundary of the group is a model for the Poisson boundary [Kaimanovich, 2000, Theorem 9.2]. This identification had been established before with the additional assumption of a finite first moment for the step distribution in [BallmannLedrappier, 1994, Theorem 2].

Another relevant family of groups are semi-simple Lie groups, which have been present in the study of Poisson boundaries since its origins. Furstenberg proved that lattices in semi-simple Lie groups admit probability measures such that the corresponding Poisson boundary is completely described by the Satake-Furstenberg boundary of the semi-simple Lie group [Furstenberg, 1967], [Furstenberg, 1971, Theorem 5.1], via the so-called "discretization" of the Brownian motion on the associated symmetric space. The Poisson boundary of **discrete subgroups of** $\mathrm{GL}(d, \mathbb{R})$ was described by [Ledrappier, 1985, Theorem A] for probability measures with a finite first moment. For general **discrete subgroups of semi-simple Lie groups**, [Kaimanovich, 1985, Theorem 5] and [Kaimanovich, 2000, Theorem 10.3 & 10.7] described the Poisson boundary for measures with finite entropy and a finite first logarithmic moment. Additionally, the Poisson boundary of $\mathrm{GL}(d, \mathbb{Q})$ is described in [BrofferioSchapira, 2011, Theorem 1.1], and the Poisson boundary of invertible triangular matrices with coefficients in a number field is described in [Schapira, 2009, Theorem 2.1].

The Poisson boundary of **mapping class groups** for probability measures with finite entropy and finite first logarithmic moment is completely described by the boundary of the Thurston compactification of the Teichmüller space [KaimanovichMasur, 1996, Theorem 2.3.1]. The Poisson boundary of **braid groups** is completely described in [FarbMasur, 1998, Theorem 4.2] (see also [MalyutinVershik, 2008, Theorem 0.3] for the stability of the so-called Markov-Ivanovsky normal form along sample paths of random walks on braid groups). The Poisson boundary of simple random walks on **right-angled Artin groups** is described in [Malyutin, 2005, Theorem 1] in terms of the stabilization of a normal form for elements of the group. In some cases, the Poisson boundary of a random walk can be identified with the κ-Morse boundary, for some appropriately chosen sublinear function $\kappa : [0, \infty) \to [1, \infty)$. This is the case of finitely supported random walks on right-angled Artin groups [QingRafi, 2022, Theorem F], mapping class groups [QingRafiTiozzo, 2020, Theorem C] and non-elementary relatively hyperbolic groups [QingRafiTiozzo, 2020, Theorem D].

For random walks on the group $\mathrm{Out}(\mathbf{F_n})$, where F_n is a free group of rank n, it is proved in [Horbez, 2016, Theorem 4.2] that the Poisson boundary of any non-elementary probability measure with finite entropy and a finite first logarithmic moment is completely described by the boundary of the free factor complex. The Poisson boundary of countable groups acting by isometries on a uniformly convex, Busemann non-positively curved, complete metric space can be identified with the visual boundary of the space [KarlssonMargulis, 1999, Corollary 6.2], with an assumption of a finite first moment on the step distribution of the random walk. In particular, this result applies to **groups acting on** $\mathrm{CAT}(0)$ **proper metric spaces** by isometries with bounded exponential growth. This result implies the description of the Poisson boundary of **Coxeter groups** through their action on the associated Moussong complex [KarlssonLedrappier, 2006, Theorem 6.1], with the same hypotheses on the measure. The Poisson boundary of certain **groups of diffeomorphisms of the circle** that satisfy some regularity conditions on their actions is described in [Deroin, 2013, Theorem 1.1]. **Free-by-cyclic groups** are studied on [GauteroMathéus, 2012, Theorem 1], where their Poisson boundary for probability measures with a finite first moment is described in terms of the boundary of the free subgroup, and they obtain similar results for other certain extensions of free and hyperbolic groups [GauteroMathéus, 2012, Theorems 2, 3 & 4]. In [CunoSava-Huss, 2018, Theorems 1.1 and A.2] it is proved that the Poisson boundary of **non-amenable Baumslag-Solitar groups** $\mathrm{BS}(p,q)$, for $p,q \in \mathbb{Z}\backslash\{-1,0,1\}$, with $|p| \neq |q|$, is described by the boundary of an infinite tree (the Bass-Serre tree associated with the group) and the boundary of the hyperbolic plane, on which these groups act. Baumslag-Solitar groups admit a natural projection to $\mathbb{Z}$, and in the non-amenable case the above description of the Poisson boundary requires a finite first moment when the induced random walk on $\mathbb{Z}$ has non-zero drift, and a finite $(2+\varepsilon)$-moment, for some $\varepsilon > 0$, when it is centered. In the case $|p| = |q|$ with $p,q \notin \{-1,0,1\}$, for which the group $\mathrm{BS}(p,q)$ is also non-amenable, the boundary is described for all step distributions with a finite first moment, in terms of the boundary of the associated Bass-Serre tree [CunoSava-Huss, 2018, Theorem A.3].

There are also classes of solvable groups for which non-trivial Poisson boundaries have been described. An example of this is the (solvable) Baumslag-Solitar group $\mathrm{BS}(1,2)$ [Kaimanovich, 1991, Theorem 4.4], for probability measures with a finite first moment. If the vertical drift is zero, the Poisson boundary is trivial. Otherwise, depending on the sign of the drift, the Poisson boundary is described by either $\mathbb{R}$ or $\mathbb{Q}_2$, endowed with the corresponding hitting measure. This description generalizes to all **solvable Baumslag-Solitar groups** $\mathrm{BS}(1,p)$, for $p \in \mathbb{Z}\backslash\{-1,0,1\}$. This result was generalized to hold for for random walks on finitely generated subgroups of the **affine group over** $\mathbb{Q}$ whose step distribution has a finite first moment, where the Poisson boundary is completely described by the product of p-adic rationals $\mathbb{Q}_p$ and $\mathbb{R}$ [Brofferio, 2006], [Brofferio, 2009, Theorem 1]. The Poisson boundary has also been described for finitely generated polycyclic groups [Kaimanovich, 1995, Theorem 1] with a finite first moment assumption, and the so-called **discrete affine group of a regular tree** [BrieusselTanakaZheng, 2021, Theorem 1.1] for finitely supported step distributions. A notable class of solvable groups for which the Poisson boundary is completely described, with assumptions on the measure μ, are wreath products $F \wr \mathbb{Z}^d$ with F a non-trivial finitely generated solvable group. We discuss these results and related descriptions of Poisson boundaries on more general wreath products in the following subsection.

3.5.1 Description of the Poisson boundary of wreath products

The exchangeability boundary of a random walk and the Poisson boundary of $\mathbb{Z}_+ \wr \mathbb{Z}^d$

The first complete description of a non-trivial Poisson boundary of a wreath product was obtained in [JamesPeres, 1996, Corollary 1.1]. They proved that the Poisson boundary of some particular degenerate probability measures on $\mathbb{Z} \wr \mathbb{Z}^d$, $d \geq 1$ is completely described by the number of visits to points of the base group $\mathbb{Z}^d$.

Definition 3.5.1. Let G be a group and μ a probability measure on G. We say that two trajectories $\mathbf{w}, \mathbf{w}' \in G^{\mathbb{Z}_+}$ are *exchangeable* if there is $N \geq 0$ such that $w_n = w'_n$ for all $n > N$ and such that the trajectory $(w_1, w_2, \ldots, w_N)$ of $\mathbf{w}$ up to instant N is a permutation of the trajectory $(w'_1, w'_2, \ldots, w'_N)$ of $\mathbf{w}'$ up to instant N.

Note that for every $g \in G$, if $\mathbf{w}, \mathbf{w}' \in G^{\mathbb{Z}_+}$ are exchangeable trajectories as in Definition 3.5.1, then for every $n > N$ the number of visits of $\mathbf{w}$ to g up to instant n is equal to the number of visits of $\mathbf{w}'$ to g up to instant n. Furthermore, if two trajectories $\mathbf{w}, \mathbf{w}' \in G^{\mathbb{Z}_+}$ coincide after some instant N and have the same number of visits to each point in G, then they are exchangeable. We define the *exchangeable boundary* of the μ-random walk as the quotient of the space of sample paths by the measurable hull of the exchangeable equivalence relation. The associated σ-algebra is called the *exchangeable σ-algebra.*

Given a probability measure μ on a group G, let us construct a probability measure $\tilde{\mu}$ on $\mathbb{Z} \wr G$ by $\tilde{\mu}((\delta_g, g)) = \mu(g)$ for $g \in G$, where $\delta_g : G \to \mathbb{Z}$ satisfies $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $h \in G \backslash \{g\}$. In other words, the $\tilde{\mu}$-random walk does a μ-random walk on the base group G, and it adds a unit increment to the lamp configuration at every visited element. Then the exchangeability boundary of the μ-random walk on G coincides with the tail boundary of the μ-random walk on $\mathbb{Z} \wr G$, which is equal to the Poisson boundary of the μ-random walk on $\mathbb{Z} \wr G$ up to $\mathbb{P}$-null sets (see Subsection 3.2.4) (see [Kaimanovich, 1991, Lemma 6.1]).

It is proved in [JamesPeres, 1996, Proposition 1.1] that if the trajectories of the μ-random walk on G almost surely has infinitely many cutpoints, then the exchangeable boundary is completely described by the number of visits to points in G. For transient simple random walks on finitely generated groups, the existence of infinitely many cutpoints is proved in [Lawler, 1991, Lemma 7.7.1] for $\mathbb{Z}^d$, $d \geq 4$, in [JamesPeres, 1996, Theorem 1.2] for all groups except for the discrete Heisenberg group $H_3(\mathbb{Z})$ and finite extensions of it, and in [Blachère, 2003, Theorem 4.1] for the latter family of groups. Hence, the combination of these results provides a complete description of the Poisson boundary of the $\tilde{\mu}$-random walk on $\mathbb{Z} \wr G$ via the number of visits to points in G, for every finitely generated group G. We remark that this had previously been proved for $G = \mathbb{Z}$ in [Kaimanovich, 1991, Theorem 6.2] for measures μ on $\mathbb{Z}$ with a finite first moment and that induce a transient random walk.

Non-degenerate random walks on wreath products

The following lemma goes back to [KaimanovichVershik, 1983], and it is the main tool used to exhibit a non-trivial μ-boundary of random walks on wreath products.

Lemma 3.5.2 (Stabilization lemma, [KaimanovichVershik, 1983, Section 6], [Kaimanovich, 1991, Theorem 3.3], [KarlssonWoess, 2007, Theorem 2.9], [Erschler, 2011, Lemma 1.1])**.** *Let μ be a probability measure on $A \wr B$ such that*

$$\sum_{(f,x)\in A\wr B} |\mathrm{supp}(f)|\mu((f,x)) < \infty.$$

Suppose that the projection of μ to B induces a transient random walk. Then for $\mathbb{P}$-almost every trajectory $\{(\varphi_n, X_n)\}_{n\geq 0}$ of the μ-random walk on $A \wr B$ and for each $b \in B$, there exists $N \geq 1$ such that for every $n \geq N$, we have $\varphi_n(b) = \varphi_N(b)$.

Since B is countable, the above implies that the lamp configuration $\{\varphi_n\}_{n\geq 0}$ stabilizes to a limit lamp configuration $\varphi_\infty : B \to A$, and that the space A^B, endowed with the corresponding hitting measure, is a μ-boundary of $A \wr B$. Note that if A and B are finitely generated, the hypotheses on μ of Lemma 3.5.2 are satisfied if μ has a finite first moment. In particular, the conclusion of Lemma 3.5.2 holds for simple random walks on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$, $d \geq 3$.

The first version of the stabilization lemma was proved for $A = \mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{Z}^d$, $d \geq 1$, for finitely supported measures in [KaimanovichVershik, 1983, Section 6], and for infinitely supported measures with a finite first moment in [Kaimanovich, 1991, Theorem 3.3]. For finite groups A and B a free group, the stabilization lemma is proved in [KarlssonWoess, 2007, Theorem 2.9] for measures with a finite first moment and bounded lamp range. Later, this result was extended to the general setting of finitely generated groups A and B in [Erschler, 2011, Lemma 1.1]. The proof of this result consists in applying the Borel-Cantelli Lemma to the sequence of events $\left\{\varphi_n(b) \neq \varphi_{n+1}(b)\right\}$, $n \geq 1$, for each $b \in B$.

For any probability measure μ on a wreath product $A \wr B$ such that the lamp configurations stabilize almost surely along the sample paths of the μ-random walk, the space of infinite lamp configurations A^B endowed with the corresponding hitting measure λ is a μ-boundary. More precisely, one considers

$$\lambda(E) := \mathbb{P}\left(\varphi_\infty \in E\right) \quad \text{for every measurable subset } E \subseteq A^B.$$

Here, we denote $\varphi_\infty : B \to A$ the random variable corresponding to the stabilized values of the lamp configuration of the μ-random walk, so that $\varphi_\infty(b) := \lim_{n\to\infty} \varphi_n(b)$ for each $b \in B$.

We now summarize the results that provide a complete description of the Poisson boundary of non-degenerate random walks on wreath products.

— The first complete description of the Poisson boundary of wreath products for non-degenerate random walks was proved by Kaimanovich using the strip criterion (Theorem 3.4.30). In [Kaimanovich, 2001, Theorem 3.6.6] he proved that the space of limit lamp configurations endowed with the hitting measure describes the Poisson boundary of the μ-random walk on $A \wr B$, supposing that

  1. μ has a finite first moment, and
  2. B is a group of subexponential growth that maps homomorphically to $\mathbb{Z}$, such that the probability measure induced by μ on $\mathbb{Z}$ has non-zero mean.

  In particular, for any finitely generated group A, the above conditions hold for random walks with a finite first moment on $A \wr \mathbb{Z}^d$, $d \geq 1$, whose projection to $\mathbb{Z}^d$ is non-centered. When $d = 1$ and μ has a finite first moment, the induced random walk on $\mathbb{Z}$ will be transient if and only if it has a non-zero mean [Spitzer, 1976, Theorem 8.1.(a)]. Hence, in the one-dimensional case, Kaimanovich's result covers all random walks with a finite first moment on $A \wr \mathbb{Z}$ with a transient projection to $\mathbb{Z}$. Similarly, when $d = 2$ and the projection of μ to $\mathbb{Z}^2$ has a finite second moment, the induced random walk on $\mathbb{Z}^2$ will be transient of and only if it has a non-zero mean [Spitzer, 1976, Theorem 8.1.(b)]. Thus, Kaimanovich's result covers in particular all measures on $A \wr \mathbb{Z}^2$ with a finite second moment and a transient projection to $\mathbb{Z}^2$. However, it is possible to find examples of transient random walks on $\mathbb{Z}^2$ with a finite first moment, an infinite second moment, and which have zero mean. Additionally, Pólya's theorem (Theorem 3.1.7) and its generalization by Varopoulos (Theorem 3.1.9) state that in dimensions $d \geq 3$, any non-degenerate random walk on $\mathbb{Z}^d$ will be transient. Hence, for every $d \geq 2$ there are non-degenerate probability measures on $A \wr \mathbb{Z}^d$ with a finite first moment, and with a projection to $\mathbb{Z}^d$ that is transient and has zero mean. Such measures are not covered by Kaimanovich's description of the Poisson boundary.

— Additional descriptions of the Poisson boundary of wreath products were obtained for base groups distinct from $\mathbb{Z}^d$. In the case where A is finite and B is a free product of finitely many copies of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ that is not virtually $\mathbb{Z}$, the Poisson boundary of $A \wr B$ is described in [KarlssonWoess, 2007, Theorem 3.2] for all probability measures with a finite first moment and bounded lamp range. The second condition means that at each step of the random walk, the lamp configuration can be modified only in a uniformly bounded neighborhood of the position in the base group. The model of the Poisson boundary that they provide is (a subspace of) the product of the space of limit lamp configurations with the geometric boundary of the Cayley graph of B with respect to standard generators (which is identified with an infinite regular tree), endowed with the corresponding hitting measure. The proof relies on Kaimanovich's strip criterion, and the construction of the strips is reminiscent of the strips used in the proof of Kaimanovich's result mentioned in the previous item. This connection is made precise in [Sava, 2010, Theorem 4.1] via the so-called "half-space method" to construct strips on wreath products, where it is used to provide a description of the Poisson boundary of $A \wr B$ for A finite and B a group with infinitely many ends, or B a non-elementary hyperbolic group [Sava, 2010, Theorems 4.2

and 4.3], with the same hypotheses as those of Karlsson and Woess.

— Erschler proved that for A finitely generated, $d \geq 5$, and for probability measures μ on $A \wr \mathbb{Z}^d$ with a finite third moment and that induce a transient random walk on $\mathbb{Z}^d$, the associated Poisson boundary is equal to the space of limit lamp configurations, endowed with the hitting measure [Erschler, 2011, Theorem 1]. Her proof consists of using the support of the limit lamp configuration together with cut-balls of the trajectory in the base group to guess which points of it are visited at certain time instants. With this, the identification of the Poisson boundary is obtained by using a version of Kaimanovich's ray criterion (Theorem 3.4.29).

— For $d \geq 3$ and probability measures μ on $A \wr \mathbb{Z}^d$ with a finite second moment and that induce a transient random walk on $\mathbb{Z}^d$, Lyons and Peres proved that the space of limit lamp configurations, endowed with the hitting measure, is the Poisson boundary of the μ-random walk on $A \wr \mathbb{Z}^d$ [LyonsPeres, 2021a, Theorem 5.1]. Additionally, they showed that for random walks on groups $A \wr B$ with A finite and B finitely generated, for step distributions with finite entropy, bounded lamp range, and that induce a transient random walk on the base group B, the associated Poisson boundary is the space of limit lamp configurations [LyonsPeres, 2021a, Theorem 1.1]. To prove these results, they estimate the number of times that the random walk on the base group can be close to locations already visited in the past by using upper bounds on the Green function. In particular for the case of $B = \mathbb{Z}^d$, the second moment assumption controls how often lamps are modified very far away from the current position. This is used to show that, for a large constant $s > 0$, it is very likely that the first n steps of the random walk on $\mathbb{Z}^d$ remain inside a ball of radius $s\sqrt{n}$ and do not modify lamps at positions of word length larger than $2s\sqrt{n}$. In addition, for a fixed $\varepsilon > 0$ there is a positive probability that after time $n(1+\varepsilon)$ the random walk on $\mathbb{Z}^d$ does not enter the ball of radius $4s\sqrt{n}$, nor that lamps at positions of word length less than $2s\sqrt{n}$ are modified after instant $n(1+\varepsilon)$. From this, they show that with positive probability the possible values for the position at time n of the random walk on $A \wr \mathbb{Z}^d$ lie in a set of size $\exp(\varepsilon n)$, and then use Kaimanovich's conditional entropy criterion to conclude the proof. Lyons and Peres also prove an enhanced version of Kaimanovich's conditional entropy criterion [LyonsPeres, 2021a, Corollary 2.3] and use it they also give a short proof of the description of the Poisson boundary for simple random walks on $A \wr \mathbb{Z}^d$, $d \geq 3$, by using cut-spheres to estimate the values of the lamp configuration at a given instant [LyonsPeres, 2021a, Theorem 3.1].

Probability measures for which the lamp configuration does not stabilize

It is proved in [Erschler, 2004b, Theorem 3.1] that any non-degenerate random walk with finite entropy on $A \wr B$ that induces a transient random walk on B has a non-trivial Poisson boundary. The proof is based on the entropy criterion (Theorem 3.4.7), and does not provide an explicit non-trivial μ-boundary for general random walks on these groups. In the results mentioned in the previous subsections, the non-triviality of the Poisson boundary of a random walk on a wreath product $A \wr B$ follows from the existence of a non-trivial μ-boundary that exists thanks to Lemma 3.5.2. However, for probability measures with an infinite first moment it may be possible that the lamp configuration does not stabilize along sample paths. This was first proved in [Kaimanovich, 1983, Proposition 1.1], where examples of probability measures on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$, that can be chosen to have a finite $(1-\varepsilon)$-moment for an arbitrary $\varepsilon > 0$, satisfy that $\varphi_n(0)$ changes infinitely often along almost every sample path. For these measures, the difference $\varphi_n(0) - \varphi_n(1)$ does stabilize to a limit value, and this provides nonetheless a non-trivial μ-boundary of the random walk. For more general examples, a phenomenon like the latter may not occur: it is proved in [Erschler, 2011, Section 6] that there are random walks with finite entropy on wreath products $A \wr B$ with a non-trivial Poisson boundary, such that there is no functional defined by a finite subset of the base group B which stabilizes along sample paths. Additionally, [LyonsPeres, 2021a, Section 5] shows that there are probability measures on $A \wr \mathbb{Z}^d$ with finite entropy for which the lamp configuration does not stabilize along sample paths, and that satisfy $\mathbb{E}[\mathrm{diam}(\mathrm{supp}(\varphi_1))^\alpha] < \infty$ if and only if $\alpha < 2$. Here $\mathrm{diam}(\mathrm{supp}(\varphi_1))$ represents the largest distance (on $\mathbb{Z}^d$) at which there is a non-trivial lamp state in the random lamp configuration φ_1, whose law is the projection of the step distribution to $\bigoplus_{\mathbb{Z}^d} A$.

3.6 The pin-down approximation approach to the identification of Poisson boundaries

All of the results regarding the identification of (non-trivial) Poisson boundaries described in Section 3.5 have in common that they require the finiteness of some moment (or logarithmic moment) of the step distribution of the random walk. In recent years, there have been results regarding the description of Poisson boundaries for larger classes of measures, with results that do not ask for the finiteness of some moment of the step distribution. These results are proved using the method of a “pin-down approximation”. In this section we will explain this approach and its applications to free semigroups in [ForghaniTiozzo, 2019], and to hyperbolic groups and acylindrically hyperbolic groups in [ChawlaForghaniFrischTiozzo, 2022].

3.6.1 The pin-down approximation

The “pin-down approximation” approach to the identification of Poisson boundaries consists of estimating the conditional entropies $H_\xi(\mathcal{A}_n)$ of the random walk by introducing additional partitions of the space of trajectories, that in average add a small amount of entropy to the process. More precisely, let G be a countable group and let μ a probability measure on G with $H(\mu) < \infty$. Let $\mathbf{X} = (X, \lambda)$ be a μ-boundary of G, and suppose that there exists countable partitions $(\rho_n)_{n\geq 1}$ of the space of trajectories $(G^{\mathbb{Z}_+}, \mathbb{P})$ that satisfy

1. $\lim_{n\to\infty} \dfrac{H(\rho_n)}{n} = 0$, and
2. $\lim_{n\to\infty} \dfrac{H_{\mathbf{X}}(\mathcal{A}_n \mid \rho_n)}{n} = 0$.

Then (X, λ) is the Poisson boundary of (G, μ). Indeed, by using the first item of Lemma 3.4.19 we have that

$$H_{\mathbf{X}}(\mathcal{A}_n) = H_{\mathbf{X}}(\rho_n) + H_{\mathbf{X}}(\mathcal{A}_n \mid \rho_n).$$

Next, we can use the fact that $H_{\mathbf{X}}(\rho_n) \leq H(\rho_n)$ (Lemma 3.4.17) to obtain

$$H_{\mathbf{X}}(\mathcal{A}_n) = H_{\mathbf{X}}(\rho_n) + H_{\mathbf{X}}(\mathcal{A}_n \mid \rho_n) \leq H(\rho_n) + H_{\mathbf{X}}(\mathcal{A}_n \mid \rho_n).$$

Dividing by n and taking the limit as n goes to infinity, the above implies that

$$\lim_{n\to\infty} \frac{H_{\mathbf{X}}(\mathcal{A}_n)}{n} = 0.$$

Thanks to Theorem 3.4.27, we conclude that (X, λ) is the Poisson boundary of (G, μ).

3.6.2 The Poisson boundary of free semigroups

We now explain how the pin-down approximation approach was used in [ForghaniTiozzo, 2019] to identify the Poisson boundary of random walks on non-abelian free semigroups for all probability measures with finite entropy, and additionally also for some particular measures with infinite entropy.

Let F_2^+ be the free semigroup of rank two. Consider a free generating $\{a, b\}$, so that F_2^+ is identified with the set of finite words in the alphabet $\{a, b\}$ together with the empty word ε, and where the semigroup operation is concatenation of words. There is a natural geometric boundary ∂F_2^+ formed by the set of infinite words in the alphabet $\{a, b\}$. Then for any probability measure μ on F_2^+ with $\mu(\varepsilon) < 1$, every sample path of the μ-random walk converges to an element of ∂F_2^+. Hence the space (F_2^+, λ), where λ is the corresponding hitting measure, is a μ-boundary of F_2^+.

The main result of [ForghaniTiozzo, 2019] is the following.

Theorem 3.6.1 ([ForghaniTiozzo, 2019, Theorem 1.2])**.** *Let μ be a non-degenerate probability measure on a free semigroup F of finite or countable rank, and denote by $(\partial F, \lambda)$ the boundary of infinite words on the generators, endowed with the hitting measure. Suppose that μ has either finite entropy, or a finite first logarithmic moment. Then $(\partial F, \lambda)$ is the Poisson boundary of (F, μ).*

This result continues previous work of Forghani and Kaimanovich, who identified the Poisson boundary of free semigroups for probability measures with a finite first logarithmic moment [ForghaniKaimanovich, 2015] (see also [Forghani, 2015, Theorem 3.6.3]). We will now explain the idea of the proof of this result under the assumption of finite entropy, using as an example the free semigroup of rank two F_2^+.

The key fact used by Forghani and Tiozzo to prove Theorem 3.6.1 is the following.

Fact: An element $g \in F_2^+$ is completely determined by its word length together with knowing one infinite word for which it is a prefix.

Let us fix μ a probability measure on F_2^+ with finite entropy, and denote by $\{w_n\}_{n\geq 1}$ the μ-random walk on F_2^+. Let $\varphi : F_2^+ \to \mathbb{N}$ the semigroup homomorphism that maps each element of F_2^+ to its word length, and define the partitions φ_k, $k \geq 1$, of the space of sample paths

$(F_2^+)^{\mathbb{Z}_+}$ by saying that two trajectories w, w' belong to the same element of φ_k if and only if $\varphi(w_k) = \varphi(w'_k)$.

The fact mentioned in the previous paragraph implies that for each infinite word $\xi \in \partial F_2^+$, we have $H_\xi(\mathcal{A}_n \mid \varphi_n) = 0$. Furthermore, since φ is a homomorphism to $\mathbb{N} \leq \mathbb{Z}$, the Choquet-Deny theorem implies that

$$\lim_{n\to\infty} \frac{H(\varphi_n)}{n} = 0.$$

From this, we see that we are in a setting where we can apply the pin-down approximation to conclude that, thanks to the conditional entropy criterion (Theorem 3.4.27), the Poisson boundary of (F_2^+, μ) is $(\partial F_2^+, \lambda)$.

3.6.3 The Poisson boundary of hyperbolic groups and acylindrically hyperbolic groups for measures with finite entropy

The first application of the pin-down approximation to non-degenerate random walks on groups was by [ChawlaForghaniFrischTiozzo, 2022], who identified the boundary of hyperbolic groups. It is a well-known fact that for any non-degenerate probability measure μ on a hyperbolic group, the sample paths of the μ-random walk on G converge almost surely to an element in the Gromov boundary ∂G. For free groups, this result goes back to [DynkinMaljutov, 1961, Section 4] simple random walks on the free generating set and to [Derriennic, 1975, Théorème 2] for finitely supported random walks. For more general probability measures, this result can be traced back to [KaimanovichVershik, 1983, Section 6.8], who also cite oral communication with Margulis for this fact, and to [CartwrightSoardi, 1989, Corollary 2.3]. For general hyperbolic groups, this result is proved in [Woess, 1993, Corollary 1] and in [Kaimanovich, 2000, Theorem 3.1.9].

Theorem 3.6.2 ([ChawlaForghaniFrischTiozzo, 2022, Theorem 1.1])**.** *Let G be a non-elementary hyperbolic group and consider μ a non-degenerate probability measure on G with finite entropy. Then the Poisson boundary of (G, μ) is the Gromov boundary ∂G endowed with the corresponding hitting measure.*

The idea of their proof is to use the fact that any boundary point $\xi \in \partial G$ determines an infinite geodesic on G, together with results of [Gouëzel, 2022] that show that a sample path of a random walk on a hyperbolic group can be decomposed into geodesic-like segments that are attached to each other along so-called “pivots”. By introducing partitions of the space of sample paths that record the distance between consecutive pivots of the trajectory, together with other additional partitions that control long time intervals that do not contain pivots, they are able to apply the pin-down approximation and use Theorem 3.4.27 to conclude that ∂G provides a model for the Poisson boundary of (G, μ).

Theorem 3.6.2 is extended in [ChawlaForghaniFrischTiozzo, 2022, Theorem 1.2] for acylindrically hyperbolic groups. In particular, this description covers many classes of groups, some of which where mentioned in Subsection 3.5. Namely, it includes mapping class groups, $\mathrm{Out}(F_n)$, irreducible right-angled Artin groups, relatively hyperbolic groups, and fundamental groups of hyperbolic manifolds. We refer to [ChawlaForghaniFrischTiozzo, 2022, Corollary 1.3] for the precise actions that are considered in each family of groups, together with the corresponding models for the Poisson boundary.

Chapter 4

The Poisson boundary of lampshuffler groups

This chapter corresponds to the preprint [Silva, 2023b].

Abstract

We study random walks on the lampshuffler group $\mathrm{FSym}(H) \rtimes H$, where H is a finitely generated group and $\mathrm{FSym}(H)$ is the group of finitary permutations of H. We show that for any step distribution μ with a finite first moment that induces a transient random walk on H, the permutation coordinate of the random walk almost surely stabilizes pointwise. Our main result states that for $H = \mathbb{Z}$, the above convergence completely describes the Poisson boundary of the random walk $(\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}, \mu)$.

Contents

4.1 Introduction

We study random walks on the semi-direct product $\mathrm{FSym}_{\mathrm{ext}}(H) \coloneqq \mathrm{FSym}(H) \rtimes H$, where H is an infinite countable group and $\mathrm{FSym}(H)$ denotes the group of bijections from H to H that coincide with the identity map outside of a finite set. Here, the action of an element $h \in H$ on a permutation $f \in \mathrm{FSym}(H)$ is defined as $(h \cdot f)(x) = hf(h^{-1}x)$, for $x \in H$. These groups are referred to as *lampshuffler groups*[1] in [BonnetGenietTesseraThomassé, 2022; GenevoisTessera,

1. The name "lampshuffler" seems to have first appeared in [GheysensMonod, 2022], and it is used there to refer to the group $\mathrm{FAlt}(H) \rtimes H$, where $\mathrm{FAlt}(H)$ is the group of finitary even permutations of H. In this paper we use the name lampshuffler for the group $\mathrm{FSym}(H) \rtimes H$, following [BonnetGenietTesseraThomassé, 2022; GenevoisTessera, 2024].

2024; GheysensMonod, 2022] due to their resemblance to lamplighter groups, and random walks on them are called mixer chains in [Yadin, 2009]. In Section 4.2 we describe the basic geometric and algebraic structure of the group $\mathrm{FSym}_{\mathrm{ext}}(H)$, and explain that it inherits the properties of H being finitely generated, amenable, or elementary amenable (Lemma 4.2.1).

Let G be a countable group and let μ be a probability measure on G. The *(right) random walk* (G, μ) is the Markov chain with state space G and with transition probabilities $p(g, h) = \mu(g^{-1}h)$, for $g, h \in G$. We assume that the random walk starts at the identity element $e_G \in G$. Random walks on lampshuffler groups have been studied in the literature. It is shown in [Yadin, 2009] that the drift function of the simple random walk for the standard generating set on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ is asymptotically equivalent to $n^{3/4}$. In [ErschlerZheng, 2020, Corollary 1.4] it is proved that the Følner function of $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$, $d \geq 1$, is asymptotically equivalent to n^{n^d}, and the return probability $\mu^{2n}(e)$ of the simple random walk is shown to be asymptotically $\exp\left(-n^{\frac{d}{d+2}} \log^{\frac{2}{d+2}} n\right)$. Given a random walk (G, μ) and denoting by $h(\mu)$ its Avez asymptotic entropy (see Subsection 4.3.4 for the definition), the problem of "full realization" consists on realizing each number in the interval $[0, h(\mu)]$ as the Furstenberg entropy of some ergodic (G, μ)-space. In [HartmanYadin, 2018, Theorem 1.4] it is proved that if H is a finitely generated nilpotent group, then the lampshuffler group $\mathrm{FSym}_{\mathrm{ext}}(H)$ has full realization. Lampshuffler groups also appear in [FeldheimSodin, 2022], where the "umpteen operator" is introduced as a representation-theoretic analog of a random Schrödinger operator, and the property of having Lifshitz tails is linked to the decay of the return probability of the simple random walk on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$.

The *Poisson boundary* of a random walk (G, μ) is a measure space that encodes the asymptotic behavior of the process. It can be defined as the space of ergodic components of the shift map in the space of infinite trajectories. There are several other equivalent definitions of the Poisson boundary of a random walk, and we recall some of them in Section 4.3. In the last decades, there has been extensive research focused on the identification of Poisson boundaries, i.e., the problem of exhibiting an explicit measure space that coincides with the Poisson boundary up to a G-equivariant measurable isomorphism. The main result of the current paper is a complete description of the Poisson boundary of $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$, for measures μ with a finite first moment that induce a transient random walk on $\mathbb{Z}$.

Let μ be a probability measure on $\mathrm{FSym}_{\mathrm{ext}}(H)$ and consider the trajectory of the μ-random walk $(F_n, S_n) \in \mathrm{FSym}_{\mathrm{ext}}(H)$, $n \geq 0$, where $F_n \in \mathrm{FSym}(H)$ is a finitely supported permutation of H and $S_n \in H$. We refer to $\{F_n\}_{n \geq 1}$ as the *permutation coordinate* of the μ-random walk.

Our first result is the following *stabilization lemma*.

Lemma 4.1.1. *Let H be a finitely generated group, and consider a probability measure μ on $\mathrm{FSym}_{\mathrm{ext}}(H)$. Suppose that μ has a finite first moment and that it induces a transient random walk on H. Then for any $h \in H$, the values $F_n(h)$, $n \geq 0$, of the permutation coordinate of the random walk almost surely stabilize to a limit value $F_\infty(h)$.*

We prove this result in Section 4.4 in a more general form, where H is not assumed to be finitely generated (Lemma 4.4.5). If we furthermore suppose that μ is non-degenerate (i.e., that supp(μ) generates $\mathrm{FSym}_{\mathrm{ext}}(H)$ as a semigroup), then the stabilization lemma shows that the Poisson boundary of $(\mathrm{FSym}_{\mathrm{ext}}(H), \mu)$ is non-trivial (Corollary 4.4.6). In particular, the Poisson boundary of any simple random walk on $\mathrm{FSym}_{\mathrm{ext}}(H)$, for H not virtually $\mathbb{Z}$ nor virtually $\mathbb{Z}^2$, is non-trivial. In contrast, simple random walks on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$ for $d = 1, 2$ have a trivial Poisson boundary (see Section 4.6). The well-known open "stability problem" asks whether the non-triviality of the Poisson boundary for a simple random walk on a finitely generated group depends on the choice of generating set. Corollary 4.4.6 together with Propositions 4.6.1 and 4.6.2 imply that for the family of groups $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$, $d \geq 1$, there is no such dependence.

It is a result of Rosenblatt [Rosenblatt, 1981, Theorem 1.10] and Kaimanovich and Vershik [KaimanovichVershik, 1983, Theorem 4.4] that every amenable group admits a probability measure with a trivial Poisson boundary. Hence, if H is infinite, amenable, and is not virtually $\mathbb{Z}$ nor virtually $\mathbb{Z}^2$, the group $\mathrm{FSym}_{\mathrm{ext}}(H)$ admits symmetric non-degenerate random walks with a transient projection to H, for which the permutation coordinate does not stabilize (Remark 4.4.7). In Proposition 4.4.8, we prove that $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ also admits random walks with this property. The stabilization lemma excludes measures as above via the assumption of a finite first moment. In Proposition 4.4.9 we show that this condition cannot be weakened to the finiteness of a smaller moment: we construct a probability measure on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ that induces a transient random walk on $\mathbb{Z}$ and that has a finite $(1-\varepsilon)$-moment, for every $0 < \varepsilon < 1$, for which the permutation coordinate does not stabilize.

We now state our main theorem.

Theorem 4.1.2. *Let μ be a probability measure on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ with a finite first moment that induces a transient random walk on $\mathbb{Z}$. Then the Poisson boundary of $(\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}), \mu)$ coincides with the space of limit functions $F_\infty : \mathbb{Z} \to \mathbb{Z}$, endowed with the corresponding hitting measure.*

We prove this result by using Kaimanovich's Conditional Entropy Criterion [Kaimanovich, 2000, Theorem 4.6]. Another component of our proof is the *displacement* associated with a permutation (Definition 4.2.2). We explain the idea of the proof of Theorem 4.1.2 at the beginning of Section 4.5, and present the proof in Subsection 4.5.1. We mention that the conditional entropy criterion, together with the Ray criterion and Strip criterion that follow from it [Kaimanovich, 2000], has played a role in the identification of the Poisson boundary for many classes of groups, some of which we mention below.

The Poisson boundary has been described for several families of countable groups, many of which possess hyperbolic-like properties. One such family is that of non-abelian free groups, whose Poisson boundary was described in [DynkinMaljutov, 1961] for measures supported on a free generating set and in [Derriennic, 1975] for finitary measures. More generally, the Poisson boundary of a (non-elementary) hyperbolic group was shown to coincide with its Gromov boundary in [Ancona, 1987] for finitary measures and in [Kaimanovich, 2000, Theorems 7.4 and 7.7] for more general μ. Furthermore, [ChawlaForghaniFrischTiozzo, 2022] established that this description of the Poisson boundary holds for any measure of finite entropy on a hyperbolic group, and described the boundary for acylindrically hyperbolic groups, extending [MaherTiozzo, 2021, Theorem 1.5]. These papers cover the description of the Poisson boundary for various classes of groups that had already been studied, with extra conditions on the measure, such as groups with infinitely many ends [Woess, 1989], [Kaimanovich, 2000, Theorem 8.4], mapping class groups [KaimanovichMasur, 1996], braid groups [FarbMasur, 1998], groups acting on $\mathbb{R}$-trees [Gautero-Mathéus, 2012] and $\mathrm{Out}(F_n)$ [Horbez, 2016]. Another family of groups we mention is that of discrete subgroups of semi-simple Lie groups, studied in [Furstenberg, 1971; Ledrappier, 1985] and [Kaimanovich, 2000, Theorems 10.3 and 10.7].

The Poisson boundary has also been described for classes of groups that do not exhibit a hyperbolic nature. A notable family is that of amenable groups, which always admit a non-degenerate measure with a trivial Poisson boundary [KaimanovichVershik, 1983; Rosenblatt, 1981]. A natural question is whether *every* non-degenerate measure on a given amenable group G has a trivial Poisson boundary. In such a case, G is called a *Choquet-Deny group* and this family of groups includes abelian groups [Blackwell, 1955; ChoquetDeny, 1960; DoobSnell-Williamson, 1960], nilpotent groups [DynkinMaljutov, 1961; Margulis, 1966], and groups that have no ICC [2] quotient [Jaworski, 2004; LinZaidenberg, 1998]. It is proved in [FrischHartman-TamuzVahidi Ferdowsi, 2019] that the latter property is also necessary and thus provides an algebraic characterization of countable Choquet-Deny groups. This result is further developed in [ErschlerKaimanovich, 2023], where the authors prove the following: any countable group G with an ICC quotient admits a non-degenerate symmetric measure of finite entropy, for which the Poisson boundary can be completely described in terms of the convergence of sample paths to the boundary of a locally finite forest, whose vertex set is G.

Kaimanovich and Vershik [KaimanovichVershik, 1983, Proposition 6.4] provided the first examples of amenable groups that admit measures with a non-trivial Poisson boundary. Namely, they proved that for the wreath product $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$, $d \geq 1$, and for any non-degenerate finitely supported measure μ whose projection to $\mathbb{Z}^d$ induces a transient random walk, the lamp configurations stabilize almost surely. This implies the non-triviality of the associated Poisson boundary, and it was conjectured that this space is completely described by the space of limit configurations. This was initially proved for measures with a projection to $\mathbb{Z}^d$ with non-zero drift [Kaimanovich, 2001, Theorem 3.6.6], whereas the case of zero drift was proved first for $d \geq 5$ [Erschler, 2011], and later for $d \geq 3$ by Lyons and Peres [LyonsPeres, 2021a]. Their proofs can be adapted to provide an analogous description of the Poisson boundary of random walks on free metabelian groups [Erschler, 2011; LyonsPeres, 2021a] and, in both cases, generalize to infinitely supported measures with appropriate moment conditions. Additional results about the description of the Poisson boundary on wreath products have been obtained in [KarlssonWoess, 2007; Sava, 2010] and we recall them in Subsection 4.3.3. Another class of amenable groups for which the Poisson boundary has been completely described (for finitely supported measures) is the family of discrete affine groups of a regular tree, studied in [BrieusselTanakaZheng, 2021].

Theorem 4.1.2 provides a new class of groups for which we have a complete description of the Poisson boundary. Since our result does not ask for non-degeneracy of μ, and the group $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ contains subgroups isomorphic to $F \wr \mathbb{Z}$ for every finite group F (Proposition 4.2.4), Theorem 4.1.2 includes the description of the Poisson boundary for the wreath product of a

2. A countable group is said to be *ICC* if it is non-trivial and every non-trivial element has an infinite conjugacy class.

finite group with an infinite cyclic base group.

In Subsection 4.2.3 we comment on the similarities and differences between the group $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ and wreath products of the form $F \wr \mathbb{Z}$, where F is finite or $F = \mathbb{Z}$. One distinction we remark is that $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ admits (degenerate) measures with non-trivial Poisson boundary that induce a recurrent random walk on $\mathbb{Z}$, whereas this is not possible for wreath products $F \wr \mathbb{Z}$ as above. Nonetheless, in Section 4.6 we show that such examples cannot be found among finitely supported measures: we prove that any finitary measure on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ or on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^2)$ which induces a recurrent random walk on the base group has a trivial Poisson boundary (Propositions 4.6.1 and 4.6.2).

There is also a difference between $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ and wreath products in terms of the relation between the stabilization of configurations and the non-triviality of the Poisson boundary. In the case of $F \wr \mathbb{Z}$, for F finite or $F = \mathbb{Z}$, and supposing that lamp configurations stabilize, the Poisson boundary will be non-trivial as soon as $\mathrm{supp}(\mu)$ contains two distinct elements with the same projection to $\mathbb{Z}$. Indeed, in such a case one can prove that there will be distinct limit configurations that occur with positive probability, which in turn provide non-trivial shift-invariant events in the space of infinite trajectories. In contrast, the analogous statement for $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ does not hold. In Example 4.4.4 we exhibit a family of virtually cyclic subgroups of $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$, generated by elements with the same projection to $\mathbb{Z}$, which has the following property: for every finitary measure with transient projection to $\mathbb{Z}$, the permutation coordinate of every trajectory stabilizes to the same limit function $F : \mathbb{Z} \to \mathbb{Z}$.

4.1.1 Organization

In Section 4.2 we define the groups $\mathrm{FSym}_{\mathrm{ext}}(H)$, show that they contain wreath products as subgroups whenever H is co-Hopfian, and comment on the similarities and differences between both families of groups. Afterward, in Section 4.3 we recall preliminary facts about random walks and Poisson boundaries. In Section 4.4 we prove the Stabilization Lemma in a more general form (Lemma 4.4.5). We prove Theorem 4.1.2 in Section 4.5. Finally, in Section 4.6 we show that finitary measures on $\mathrm{FSym}_{\mathrm{ext}}(H)$, for $H = \mathbb{Z}$ or $H = \mathbb{Z}^2$, whose projection to H induces a recurrent random walk, have a trivial Poisson boundary.

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4.2 Extensions of finitary symmetric groups

Let H be a countable group, and consider the group $\mathrm{FSym}(H)$ of bijective functions $f : H \to H$ such that only finitely many values $h \in H$ satisfy $f(h) \neq h$. We call the set of such elements the *support* of f and denote it by $\mathrm{supp}(f)$. Note that the group operation of $\mathrm{FSym}(H)$ is the composition of functions, in contrast to the direct sum $\bigoplus_H H$, where the group operation is pointwise multiplication. We will denote the identity element of $\mathrm{FSym}(H)$ by id.

Since H has its own group structure, there is a natural left action of H on functions $f : H \to H$. More precisely, for every $h \in H$ and $f : H \to H$ we define $h \cdot f$ by

$$(h \cdot f)(x) = hf(h^{-1}x), \text{ for } x \in H.$$

Whenever f has finite support, so does $h \cdot f$ and one has $\mathrm{supp}(h \cdot f) = h \cdot \mathrm{supp}(f)$. Similarly, if f is a bijection, then so is $h \cdot f$. With this, H has a well-defined action on $\mathrm{FSym}(H)$, and we can consider the semi-direct product $\mathrm{FSym}_{\mathrm{ext}}(H) \coloneqq \mathrm{FSym}(H) \rtimes H$.

We now mention some of the geometric properties of these groups that have been studied in the past decades, in addition to the ones related to random walks that were already discussed in the introduction. The lampshuffler group $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ is considered in [VershikGordon, 1997, Section 2.3] as an example of a finitely generated group that is locally embeddable in the class of finite groups (LEF) that is not residually finite (in the same paper it is mentioned that this example goes back to Vershik's Doctor of Sciences thesis [Vershik, 1973]). Additionally, the group $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ was used in [Stëpin, 1983] as an example of a finitely generated non-residually finite group which admits a freely approximable action. This is in contrast with [Stëpin, 1983, Theorem 1], which states that any finitely presented group that admits such an action must be residually finite. The subgroup $\mathrm{FAlt}(\mathbb{Z}) \rtimes \mathbb{Z}$ of the lampshuffler group $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$, where $\mathrm{FAlt}(\mathbb{Z})$ stands for the group of finitary even permutations of $\mathbb{Z}$, is used in [Chou, 1980, Example 2] as an example of the existence of free subsemigroups in elementary amenable groups that are not virtually solvable. In [ElekSzabó, 2006, Theorem 3] it is shown that if H is an infinite, hyperbolic, residually finite group with Kazhdan's property (T), then the group $\mathrm{FSym}_{\mathrm{ext}}(H)$ is sofic but not residually amenable. The group $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ is used in [BrieusselZheng, 2019, Proposition 4.4] to provide an example of a locally-finite-by-$\mathbb{Z}$ group that does not possess Shalom's property H_{FD}.

4.2.1 Basic properties

Note that the group $\mathrm{FSym}(H)$ is locally finite, meaning that every finitely generated subgroup is finite, and hence it is elementary amenable. Since (elementary) amenability is preserved by group extensions, the group $\mathrm{FSym}_{\mathrm{ext}}(H)$ is (elementary) amenable whenever H is.

We now show that whenever H is finitely generated then so is the group $\mathrm{FSym}_{\mathrm{ext}}(H)$, and exhibit an explicit standard choice of generators.

Define for any $x, y \in H$ the transposition $\delta_x^y \in \mathrm{FSym}(H)$ by

$$\delta_x^y(h) = \begin{cases} y, & \text{if } h = x, \\ x, & \text{if } h = y, \text{ and} \\ h & \text{otherwise.} \end{cases}$$

That is, δ_x^y corresponds to the bijection of H that swaps x and y, while leaving the rest of H unchanged. We denote $\delta_x \coloneqq \delta_x^{e_H}$, for $x \in H$.

Suppose now that H is finitely generated, and fix a finite generating set S_H of H. Then a standard generating set for $\mathrm{FSym}_{\mathrm{ext}}(H)$ is given by $S_{\mathrm{std}} = S_H \cup \{\delta_s \mid s \in S\}$. Indeed, it suffices to note that conjugating δ_s by generators of S_H allows one to obtain transpositions between adjacent elements of any arbitrarily large ball of H, and thus any permutation supported on it.

Note that for any $(f, x) \in \mathrm{FSym}_{\mathrm{ext}}(H)$, we have $(f, x) \cdot (\mathsf{id}, h) = (f, xh)$, for $h \in H$, as well as $(f, x) \cdot (\delta_s, e_H) = (f \circ (x \cdot \delta_s), x) = (f \circ \delta_{xs}^x, x)$, for $s \in S$. Hence, multiplying by elements of H corresponds to a translation of the second coordinate, while multiplying by δ_s corresponds to precomposing the first coordinate by a transposition in the current position x in the direction of s.

We summarize the above discussion in the following lemma.

Lemma 4.2.1. *Let H be a countable group, and consider the extension* $\mathrm{FSym}_{\mathrm{ext}}(H) \coloneqq \mathrm{FSym}(H) \rtimes H$.

1. *If H is finitely generated by S_H, then $\mathrm{FSym}_{\mathrm{ext}}(H)$ is also finitely generated and the set $S_{\mathrm{std}} \coloneqq S_H \cup \{\delta_s \mid s \in S_H\}$ is a finite generating set.*
2. *If H is amenable, then so is $\mathrm{FSym}_{\mathrm{ext}}(H)$.*
3. *If H is elementary amenable, then so is $\mathrm{FSym}_{\mathrm{ext}}(H)$.*

The following quantity will appear in the proof of Theorem 4.1.2.

Definition 4.2.2. Given a word metric d_H on H, we define the *displacement* of a permutation $\sigma \in \mathrm{FSym}(H)$ by $\mathrm{Disp}(\sigma) \coloneqq \sum_{h \in H} d_H(h, \sigma(h))$.

Note that the above is well defined: since σ is finitely supported, the values $d_H(h, \sigma(h))$ vanish for all but finitely many elements $h \in H$. We have the following lower bound for the word length in $\mathrm{FSym}_{\mathrm{ext}}(H)$ with respect to the generating set S_{std}.

Lemma 4.2.3. *For every $(\sigma, x) \in \mathrm{FSym}_{\mathrm{ext}}(H)$,*

$$\|(\sigma, x)\|_{S_{\mathrm{std}}} \geq \max\left\{\frac{1}{2}\mathrm{Disp}(\sigma), \|x\|_H\right\}.$$

Proof. Since the only elements of S_{std} that have non-trivial projection to H are those of S_H, it holds that $\|(\sigma, x)\|_{S_{\mathrm{std}}} \geq \|x\|_H$. Indeed, the above implies that any geodesic word of length n for (σ, x) in S_{std} projects to a word of length n in S_H that evaluates to x in H.

On the other hand, each multiplication by a transposition $\delta_s \in S_{\mathrm{std}}$ changes the value of $\mathrm{Disp}(\cdot)$ by at most 2 units. This implies that $\mathrm{Disp}(\sigma) \leq 2\|(\sigma, x)\|_{S_{\mathrm{std}}}$. □

4.2.2 Wreath products subgroups in lampshufflers

Given groups A and B, we recall that their wreath product is defined by $A \wr B := \bigoplus_B A \rtimes B$ (see also Subsection 4.3.3). As we mentioned in the introduction, it is natural to compare $\mathrm{FSym}_{\mathrm{ext}}(H)$ to wreath products of the form $F \wr H$, for F a finite non-trivial group. The objective of this subsection is to show a condition that guarantees that $\mathrm{FSym}_{\mathrm{ext}}(H)$ contains a subgroup isomorphic to a wreath product $F \wr H$ with F an arbitrary finite group. In particular, we will see that this holds for $H = \mathbb{Z}^d$, $d \geq 1$, as well as for any free group.

Recall that a group H is called *co-Hopfian* if every injective endomorphism of H is an isomorphism. In other words, a group is co-Hopfian if and only if it is not isomorphic to a proper subgroup of itself.

Proposition 4.2.4. *Let H be an infinite group that is not co-Hopfian. Then for every finite group F, the group $\mathrm{FSym}_{\mathrm{ext}}(H)$ contains a subgroup isomorphic to $F \wr H$.*

Proof. Suppose that $K \leqslant H$ is a proper subgroup with $K \cong H$ and fix $n \geq 1$. Then we can choose K such that its index $[H : K]$ is some value $m \in \{n, n+1, \ldots\} \cup \{\infty\}$. Indeed, if $\varphi : H \to H$ is an injective homomorphism then $[H : \varphi^n(H)] = [H : \varphi(H)]^n$. The subgroup generated by K together with the (finite) symmetric group on a section of H/K generates $\mathrm{FSym}(m) \wr K \cong \mathrm{FSym}(m) \wr H$, where $\mathrm{FSym}(m)$ is the group of finitary permutations on a set of cardinality m. Since any finite group embeds into $\mathrm{FSym}(m)$ for some $m \geq 2$, this shows the proposition. □

Examples of groups that are not co-Hopfian are free abelian groups, free groups, solvable Baumslag-Solitar groups [Harpe, 2000, Section III.22] and Thompson's group F [Wassink, 2011]. In contrast, the family of co-Hopfian groups includes lattices in semi-simple Lie groups [Prasad, 1976], one-ended torsion-free hyperbolic groups [Sela, 1997], the group $\mathrm{Out}(F_n)$, $n \geq 3$ [Farb-Handel, 2007; HorbezWade, 2020], and the mapping class group of a closed hyperbolic surface [IvanovMcCarthy, 1999].

In the case of $H = \mathbb{Z}^d$, $d \geq 1$, we can show that the embedding of a wreath product into $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$ does not distort its word metric. Recall that given two metric spaces (X, d_X) and (Y, d_Y), the space (X, d_X) *embeds quasi-isometrically* into (Y, d_Y) if there exists a function $f : X \to Y$ and constants $C \geq 0$, $K \geq 1$ such that for every $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C.$$

Proposition 4.2.5. *For any $d \geq 1$ and any (non-trivial) finite group F, the wreath product $F \wr \mathbb{Z}^d$ embeds quasi-isometrically into $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$.*

Proof. Let F be any non-trivial and finite group. Consider $m \geq 2$ such that F is isomorphic to a subgroup of $\mathrm{Sym}(m)$ the symmetric group on m elements. Since $F \wr \mathbb{Z}^d$ embeds quasi-isometrically into $\mathrm{Sym}(m) \wr \mathbb{Z}^d$, in order to prove the proposition it suffices to show that $\mathrm{Sym}(m) \wr \mathbb{Z}^d$ embeds quasi-isometrically into $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$.

Throughout the proof we will consider $\mathbb{Z}^d$ and its finite index subgroup $H := (m\mathbb{Z})^d$. In what follows we will use additive notation when referring to the group operation of $\mathbb{Z}^d$ or H.

Let us denote $T \coloneqq \{0, 1, \ldots, m-1\}^d \subseteq \mathbb{Z}^d$, which corresponds to a set of representatives for $\mathbb{Z}^d/H$. Then $\mathbb{Z}^d$ is partitioned into the cosets $\mathbf{h} + T$, for $\mathbf{h} \in H$. This implies that we have the embedding of $\mathrm{Sym}(m) \wr \mathbb{Z}^d \cong \mathrm{Sym}(T) \wr H \leqslant \mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$. In order to show that this is a quasi-isometric embedding, let us introduce finite generating sets for these groups.

Consider $S_{\mathbb{Z}^d} = \{\pm\hat{\mathbf{e}}_1, \pm\hat{\mathbf{e}}_2, \ldots, \pm\hat{\mathbf{e}}_d, \mathbf{0}\}$ the canonical generating set of $\mathbb{Z}^d$ together with the identity element $\mathbf{0} \in \mathbb{Z}^d$. Consider the generating set $S_H = \{\pm m\hat{\mathbf{e}}_1, \pm m\hat{\mathbf{e}}_2, \ldots, \pm m\hat{\mathbf{e}}_d, \mathbf{0}\}$ for H. With this, we define the finite generating sets

$$S_{\mathrm{wreath}} = \{(\sigma, s) \mid \mathrm{Sym}(T) \text{ and } s \in S_H\}$$

for $\mathrm{Sym}(T) \wr H$, and

$$S_{\mathrm{ext}} = \{(\sigma, s) \mid \mathrm{Sym}(T) \text{ and } s \in S_{\mathbb{Z}^d} \cup S_H\}$$

for $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$. Note that $S_{\mathrm{wreath}} \subseteq S_{\mathrm{ext}}$, and hence that $\|g\|_{S_{\mathrm{ext}}} \leq \|g\|_{S_{\mathrm{wreath}}}$ for all $g \in \mathrm{Sym}(T) \wr H$. In order to finish the proof, it suffices to show the existence of a constant $C > 0$ such that for every $g \in \mathrm{Sym}(T) \wr H$, we have

$$\|g\|_{S_{\mathrm{wreath}}} \leq C\|g\|_{S_{\mathrm{ext}}}. \tag{4.1}$$

Let us consider an arbitrary element $g = (f, x) \in \mathrm{Sym}(T) \wr H$, and denote $n \coloneqq \|g\|_{S_{\mathrm{ext}}}$.

Consider a word $w_1 w_2 \cdots w_n$ with $w_i \in S_{\mathrm{ext}}$ for each $i = 1, \ldots, n$ and $g = w_1 w_2 \cdots w_n$. For every $i = 1, \ldots, n$, define $\mathbf{h}_i \in H$ to be the unique element such that the projection to $\mathbb{Z}^d$ of $w_1 \cdots w_i$ belongs to $\mathbf{h}_i + T$. Also denote $\mathbf{h}_0 \coloneqq \mathbf{0} \in \mathbb{Z}^d$. In particular, we have $\mathbf{h}_n = x$, which corresponds to the projection of g to $\mathbb{Z}^d$. We remark additionally that for every $i = 0, 1, \ldots, n-1$, we have that $h_{i+1} = h_i + \sum_{j=1}^{d} \varepsilon_j m\hat{\mathbf{e}}_j$ for some values $\varepsilon_j \in \{-1, 0, 1\}$, for $j = 1, \ldots, d$.

Note that for every $\mathbf{h} \in H$, the permutation f restricts to a bijection $f : \mathbf{h} + T \to \mathbf{h} + T$, and that

$$\mathrm{supp}(f) \subseteq \bigcup_{i=0}^{n} (\mathbf{h}_i + B + T),$$

where $B = \left\{\sum_{i=1}^{d} \varepsilon_i m\hat{\mathbf{e}}_i \mid \varepsilon_1, \ldots, \varepsilon_d \in \{0, 1\}\right\}$. This is since the function f can only act non-trivially on copies of T that correspond to $\mathbf{h}_i + T$, $i = 0, 1, \ldots, n$, or that are neighboring to one of these cosets.

Then, since the generating set S_{wreath} allows to have any permutation in $\mathrm{Sym}(T)$ accompanying any generator $s \in S_H$, in order to show Equation (4.1) it suffices to prove the following statement: there is a constant $C > 0$, that only depends on d, such that there is a path of length at most Cn on the Cayley graph of H with respect to S_H, that starts at $\mathbf{0} \in H$, finishes at $x \in H$ and which visits all elements in the set

$$\{\mathbf{h}_i + b \mid i = 0, 1, \ldots, n \text{ and } b \in B\}.$$

For every $i = 0, 1, \ldots, n-1$ we consider a path p_i on the Cayley graph of H with respect to S_H (whose vertices are identified with the lattice $(m\mathbb{Z})^d \subseteq \mathbb{Z}^d$) that begins at $\mathbf{h}_i$, finishes at $\mathbf{h}_{i+1}$ and visits all elements in the set $\{\mathbf{h}_i + b \mid b \in B\}$. The length of such a path can be chosen to be less than or equal to $2d \cdot 2^d + d$. Indeed, the path p_i can be formed as follows: consider the cycles that go from $\mathbf{h}_i$ to $\mathbf{h}_i + b$ and back to $\mathbf{h}_i$ for each $b \in B$, which each has length at most $2d$, and there are 2^d elements in B. By concatenating these cycles together with a path from $\mathbf{h}_i$ to $\mathbf{h}_{i+1}$, which can be chosen to have length at most d, we obtain the path p_i of length at most $2d \cdot 2^d + d$. Finally, we concatenate all paths p_i, $i = 0, 1, \ldots, n$, and obtain a path of length at most $(2d \cdot 2^d + d)n$ on the Cayley graph of H with respect to S_H, that starts at $\mathbf{0} \in H$, finishes at $x \in H$ and which visits all elements in the set

$$\{\mathbf{h}_i + b \mid i = 0, 1, \ldots, n \text{ and } b \in B\}.$$

This path can then be used, as explained two paragraphs above, to construct a word of length at most $(2d \cdot 2^d + d)n$ using generators from S_{wreath} that evaluates as a group element to g. This finishes the proof, since we have proved that one can choose $C = 2d \cdot 2^d + d$ so that Equation (4.1) holds. □

4.2.3 Cyclic extensions of locally finite groups

Our main theorem (Theorem 4.1.2) concerns the group $\mathrm{FSym}_{\text{ext}}(\mathbb{Z})$, which is an extension by $\mathbb{Z}$ of the locally finite group $\mathrm{FSym}(\mathbb{Z})$. In this subsection, we discuss the geometric properties present in groups that exhibit this algebraic structure and motivate the study of random walks on them.

Recall that a group $\mathcal{L}$ is called *locally finite* if every finitely generated subgroup is finite. A group G is a *cyclic extension of a locally finite group* or *locally-finite-by-$\mathbb{Z}$* if there is a short exact sequence

$$1 \to \mathcal{L} \to G \to \mathbb{Z} \to 1,$$

where $\mathcal{L}$ is a locally finite group. Observe that any such sequence must necessarily split, so that $G = \mathcal{L} \rtimes \mathbb{Z}$. In particular, the group $\mathrm{FSym}_{\text{ext}}(\mathbb{Z})$ as well as wreath products of the form $F \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} F \rtimes \mathbb{Z}$, for a non-trivial and finite group F, are cyclic extensions of locally finite groups. Other groups in this category are the discrete affine groups of a regular tree. These groups were introduced in [BrieusselTanakaZheng, 2021], where their Poisson boundary was described for finitary measures. If every finitely generated subgroup of $\mathcal{L}$ is contained in a finite *normal* subgroup of $\mathcal{L}$, then we say that $\mathcal{L}$ is *locally normally finite*. Notably, $\bigoplus_{\mathbb{Z}} F$ is locally normally finite, while $\mathrm{FSym}(\mathbb{Z})$ is not. This provides an algebraic distinction between the groups $F \wr \mathbb{Z}$ and $\mathrm{FSym}_{\text{ext}}(\mathbb{Z})$. Cyclic extensions of locally normally finite groups are studied in [BrieusselZheng, 2019].

Cyclic extensions of locally finite groups can have arbitrarily fast-growing Følner functions. This was first observed in the last remark of Section 3 in [Erschler, 2003] (see also [Gromov, 2008, Section 8.2]), and a proof of this result is given in [OlshanskiiOsin, 2013, Corollary 1.5]. In [BrieusselZheng, 2021, Theorem 1.1], it is proved that locally-finite-by-$\mathbb{Z}$ groups exhibit a large class of speed, return probability, entropy, isoperimetric profiles, and L_p-compression functions. In particular, the authors prove that any sufficiently regular function that grows at least exponentially can be realized as the Følner function of a locally-finite-by-$\mathbb{Z}$ group, which admits a simple random walk with a trivial Poisson boundary [BrieusselZheng, 2021, Corollary 4.7]. This behavior differs from what happens for the *linear algebraic Følner function*, introduced in [Gromov, 2008, Section 1.9]. In general, this function is bounded above by the usual (combinatorial) Følner function, and both of them coincide for left-orderable groups [Gromov, 2008, Section 3.2]. In contrast, the linear algebraic Følner function of every locally-finite-by-$\mathbb{Z}$ group grows linearly [Gromov, 2008, Section 8.1].

There are locally-finite-by-$\mathbb{Z}$ groups that admit simple random walks with a non-trivial Poisson boundary. Examples of this are $\mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z})$ and the discrete affine group of a regular tree [BrieusselTanakaZheng, 2021]. On the other hand, we mentioned above that [BrieusselZheng, 2021, Corollary 4.7] provides a large family of locally-finite-by-$\mathbb{Z}$ groups that admit simple random walks with a trivial Poisson boundary. As a more concrete example, simple random walks on the groups $F \wr \mathbb{Z}$, for F finite or $F = \mathbb{Z}$, and $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ also have a trivial boundary. The case of wreath products is proved in [KaimanovichVershik, 1983, Proposition 6.2], while the case of $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ follows from the sublinear asymptotics of the drift function [Yadin, 2009]. Moreover, every probability measure on $F \wr \mathbb{Z}$ with a recurrent projection to the base group $\mathbb{Z}$ has a trivial Poisson boundary. This is proved in [KaimanovichVershik, 1983, Proposition 6.3] and [Kaimanovich, 1983] for finitary measures, in [Kaimanovich, 1991] for finite first moment measures, and in [LyonsPeres, 2021a, Proposition 4.9] for the general case, by using the classification of Choquet-Deny groups [FrischHartmanTamuzVahidi Ferdowsi, 2019]. In contrast, this generalization does not hold for $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$. Indeed, the group $\mathrm{FSym}(\mathbb{Z})$ admits symmetric measures with finite entropy and with a non-trivial Poisson boundary [Kaimanovich, 1983] (this also follows from [FrischHartmanTamuzVahidi Ferdowsi, 2019], since every non-trivial element of $\mathrm{FSym}(\mathbb{Z})$ has an infinite conjugacy class). In consequence, $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ admits (degenerate) random walks with a non-trivial Poisson boundary and a recurrent projection to $\mathbb{Z}$.

4.3 Background on random walks and Poisson boundaries

4.3.1 Random walks on groups

Let G be a countable group and μ a probability measure on G. The *(right) μ-random walk* (G, μ) is the Markov chain with state space G, whose transition probabilities are given by

$$p(g, h) := \mu(g^{-1}h),\ \ g, h \in G.$$

We assume that the μ-random walk starts at the identity $e_G \in G$. The space of infinite trajectories of the random walk $G^{\mathbb{Z}_+}$ is endowed with the probability $\mathbb{P}$, which is the push-forward of the Bernoulli measure $\mu^{\mathbb{N}}$ on the space of increments $G^{\mathbb{Z}_+}$ through the map

$$\begin{aligned} G^{\mathbb{Z}_+} &\to G^{\mathbb{Z}_+} \\ (g_1, g_2, g_3, \dots) &\mapsto (g_1, g_1g_2, g_1g_2g_3, \dots). \end{aligned}$$

Suppose that G is finitely generated, and let ℓ_G be a word length on G. For $\alpha > 0$, a probability measure μ on G is said to have a *finite α-moment* if $\sum_{g\in G} \ell_G(g)^\alpha \mu(g) < \infty$. This property does not depend on the choice of ℓ_G, since changing the word length on a group modifies the metric by a multiplicative constant.

4.3.2 The Poisson boundary

Let us now recall the definition of the Poisson boundary, together with some equivalent definitions. For background on random walks on countable groups and their Poisson boundaries, we refer to [KaimanovichVershik, 1983; Kaimanovich, 2000], the surveys [Erschler, 2010; Zheng, 2023], and the introduction and Section 3 of [ErschlerKaimanovich, 2023].

Given two trajectories $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ in $G^{\mathbb{Z}_+}$, say that they are *orbit equivalent* if there exist $p, N \geq 0$ such that $x_{p+n} = y_n$ for every $n \geq N$. Consider the measurable hull of this equivalence relation in $G^{\mathbb{Z}_+}$. That is, the σ-algebra of measurable subsets of $G^{\mathbb{Z}_+}$ which are unions of the equivalence classes, modulo $\mathbb{P}$-null sets. The associated quotient of the space of infinite trajectories by this measurable hull is called the *Poisson boundary* $\partial_\mu G$ of the random walk (G, μ). Equivalently, the Poisson boundary is the space of ergodic components of the *shift map* of the space of infinite trajectories, defined by

$$\begin{aligned} T : G^{\mathbb{Z}_+} &\to G^{\mathbb{Z}_+} \\ (x_1, x_2, x_3, \ldots) &\mapsto (x_2, x_3, \ldots). \end{aligned}$$

If we do not allow the shift by p in the definition of the orbit equivalence relation above, we obtain the *tail equivalence* relation. The associated quotient space is called the *tail boundary* of the random walk, and it provides an alternative and equivalent definition of the Poisson boundary of a random walk on a group [Derriennic, 1980; KaimanovichVershik, 1983]. We remark that the Poisson boundary can be defined for general Markov chains, and these two definitions are no longer equivalent in this broader context (see [BlackwellFreedman, 1964, Example 2] and [Kaimanovich, 1992, Theorem 2.2]).

Recall that a function $f : G \to \mathbb{R}$ is called *μ-harmonic* if for every $g \in G$, it holds that $f(g) = \sum_{h\in G} f(gh)\mu(h)$. The Poisson boundary of (G, μ) can be described in terms of the space of bounded μ-harmonic functions on the subgroup generated by $\mathrm{supp}(\mu)$. In particular, the non-triviality of the Poisson boundary of a non-degenerate random walk is equivalent to the existence of non-constant bounded μ-harmonic functions on G.

Now we introduce the concept of a μ-boundary of G. Let us denote by $\mathbf{bnd} : G^{\mathbb{Z}_+} \to \partial_\mu G$ the associated quotient map from the space of trajectories onto the Poisson boundary. The space $\partial_\mu G$ is endowed with the so-called *harmonic measure* $\nu \coloneqq \mathbf{bnd}_*(\mathbb{P})$, which satisfies the equation $\mu * \nu = \nu$. One says that ν is *μ-stationary*. Thus, the Poisson boundary of (G, μ) has the structure of a measure space $(\partial_\mu G, \mathcal{F}, \nu)$ endowed with a measurable G-equivariant map $\mathbf{bnd} : G^{\mathbb{Z}_+} \to \partial_\mu G$ that satisfies

1. $\mathcal{I} = \mathbf{bnd}^{-1}(\mathcal{F})$ modulo $\mathbb{P}$-null sets, where $\mathcal{I}$ is the sub-σ-algebra of shift-invariant events of the space of trajectories $G^{\mathbb{Z}_+}$, and

2. the measure $\nu = \mathbf{bnd}_*(\mathbb{P})$ is μ-stationary, which by definition means that it satisfies the equation $\nu = \mu * \nu := \sum_{g \in G} \mu(g) g\nu$.

In general, a measure space $(B, \mathcal{A}, \lambda)$ endowed with a measurable G-action is called a *μ-boundary* of G if there exists a G-equivariant measurable map $\pi : G^{\mathbb{Z}_+} \to B$ for which $\lambda = \pi_*(\mathbb{P})$ is μ-stationary and such that $\pi^{-1}(\mathcal{A}) \subseteq \mathcal{I}$ modulo $\mathbb{P}$-null sets. The Poisson boundary of (G, μ) is the *maximal* μ-boundary of G, in the sense that for every μ-boundary $(B, \mathcal{A}, \lambda)$, the projection $\pi : G^{\mathbb{Z}_+} \to B$ factors through the map **bnd**, and it is unique up to a G-equivariant measurable isomorphism.

4.3.3 The Poisson boundary of wreath products

We illustrate the above notions with the case of wreath products, which also serves as a point of comparison with the results of this paper.

Recall that given groups A and B, their wreath product $A \wr B$ as the semidirect product $\bigoplus_B A \rtimes B$, where B acts by translations on the direct sum $\bigoplus_B A$. We remark that some authors use the notation $B \wr A$. Wreath products are also called "lamplighter groups", and for an element $(f, x) \in A \wr B$, the function f is referred to as the "lamp configuration".

Kaimanovich and Vershik [KaimanovichVershik, 1983] proved that for $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$, $d \geq 1$, and for any non-degenerate finitely supported measure μ whose projection to $\mathbb{Z}^d$ induces a transient random walk, the lamp configurations stabilize almost surely. In other words, with probability one, the values assigned by the lamp configuration to a given finite subset of $\mathbb{Z}^d$ change only finitely many times along the trajectory of the random walk. Hence, the space of limit lamp configurations has the structure of a μ-boundary, and its non-triviality implies that the Poisson boundary is non-trivial as well. These were the first examples of measures with non-trivial boundary on amenable groups, and Kaimanovich and Vershik conjectured that the space of limit configurations coincides with the Poisson boundary. This was proved under the hypotheses of a finite first moment of μ and projection to $\mathbb{Z}^d$ with non-zero drift by Kaimanovich [Kaimanovich, 2001, Theorem 3.6.6], whereas the case of zero drift, for $d \geq 3$, remained open. The Poisson boundary was later described for $A \wr B$ where A is finite and B is a free group [KarlssonWoess, 2007], or where B has infinitely many ends, or is hyperbolic [Sava, 2010]. The case of measures on $A \wr \mathbb{Z}^d$ with a centered projection to $\mathbb{Z}^d$ was proved for $d \geq 5$ by Erschler [Erschler, 2011], and by Lyons and Peres for $d \geq 3$ [LyonsPeres, 2021a], both with additional moment conditions on μ.

4.3.4 The conditional entropy criterion

Recall that the *entropy* $H(\mu)$ of a probability measure μ on G is defined as $H(\mu) := -\sum_{g \in G} \mu(g) \log(\mu(g))$. Avez [Avez, 1972] introduced the *asymptotic entropy* of the random walk (G, μ), defined as $h(\mu) := \lim_{n \to \infty} H(\mu^{*n})/n$. The existence of this limit is guaranteed by the subadditivity of the sequence $H_n := H(\mu^{*n})$, $n \geq 1$. Avez [Avez, 1974] proved that if $h(\mu) = 0$, then the random walk has a trivial Poisson boundary. Furthermore, the *Entropy Criterion*, due to Derrienic [Derriennic, 1980] and Kaimanovich and Vershik [KaimanovichVershik, 1983] states that if $H(\mu) < \infty$, then $h(\mu) = 0$ if and only if the Poisson boundary of the μ-random walk on G is trivial. This criterion was strengthened by Kaimanovich to a *Conditional Entropy Criterion*, which we state below.

Theorem 4.3.1 ([Kaimanovich, 2000, Theorem 4.6])**.** *Let μ be a probability measure on G with finite entropy, and consider $\mathbf{B} = (B, \mathcal{A}, \nu)$ a μ-boundary of G. Suppose that for every $\varepsilon > 0$ there exists a random sequence of finite subsets $\{Q_{n,\varepsilon}\}_{n\geq 1}$ of G such that*

1. *the random set $Q_{n,\varepsilon}$ is a measurable function with respect to $\mathcal{A}$, for every $n \geq 1$;*
2. $\limsup_{n\to\infty} \frac{1}{n} \log |Q_{n,\varepsilon}| < \varepsilon$ *almost surely; and*
3. $\limsup_{n\to\infty} \mathbb{P}(x_n \in Q_{n,\varepsilon}) > 0$, *where $\{x_n\}_{n\geq 1}$ is the trajectory of the μ-random walk.*

Then $\mathbf{B}$ coincides with the Poisson boundary of (G, μ).

This result is a common tool used to prove the maximality of a μ-boundary for a random walk on a group, and we will use it in the proof of Theorem 4.1.2. We mention that Lyons and Peres [LyonsPeres, 2021a, Corollary 2.3] proved an alternative version of this criterion, where the third condition of Proposition 4.3.1 is replaced by

$$\limsup_{n\to\infty} \mathbb{P}(\text{ there exists } m \geq n \text{ such that } x_m \in Q_{n,\varepsilon}) > 0.$$

This condition is easier to verify in some situations [BrieusselTanakaZheng, 2021; LyonsPeres, 2021a]. For Theorem 4.1.2 we will apply the original version.

4.4 Random walks on $\mathrm{FSym}_{\mathrm{ext}}(H)$

Let μ be a probability measure on $\mathrm{FSym}_{\mathrm{ext}}(H)$. We will use the notation $\{(\sigma_k, X_k)\}_{k\geq 1}$ for a sequence of independent increments distributed according to μ, and denote by $\{(F_n, S_n)\}_{n\geq 0}$ the trajectory of the μ-random walk on $\mathrm{FSym}_{\mathrm{ext}}(H)$. In other words, we have $(F_0, S_0) = (\mathsf{id}, e_H)$ and for $n \geq 1$,

$$(F_n, S_n) = (\sigma_1, X_1) \cdot (\sigma_2, X_2) \cdots (\sigma_n, X_n).$$

4.4.1 Stabilization of the permutation coordinate

We study conditions on the measure μ that guarantee the stabilization of the permutation coordinate to a limit function $F_\infty : H \to H$. We first give a precise definition of stabilization.

Definition 4.4.1. We say that the permutation coordinate $\{F_n\}_{n\geq 1}$ of the μ-random walk $\{(F_n, S_n)\}_{n\geq 0}$ on $\mathrm{FSym}_{\mathrm{ext}}(H)$ *stabilizes* if almost surely for every $h \in H$, there exists $N \geq 1$ such that $F_n(h) = F_N(h)$ for all $n \geq N$.

Whenever the permutation coordinate stabilizes, we can associate with almost every trajectory $\{(F_n, S_n)\}_{n\geq 0}$ of the μ-random walk a limit function $F_\infty : H \to H$, which satisfies for every $h \in H$, $F_n(h) = F_\infty(h)$ for large enough n. Since every function F_n is a bijection from H to itself, F_∞ will be injective. However, it may happen that the limit function is not surjective.

Example 4.4.2. Consider $F(X)$ the free group on a finite set $X \neq \varnothing$, and let μ be any probability measure on $\mathrm{FSym}_{\mathrm{ext}}(F(X))$ whose support is the set $\{(\delta_x, x) \mid x \in X\}$. The projection $\{S_n\}_n$ of the μ-random walk to $F(X)$ is supported on the free semigroup generated by X, and hence it will converge to a geodesic ray $\gamma : [0, +\infty) \to F(X)$. Since the support of μ consists of elements of the from (δ_x, x), the permutation coordinate of the μ-random walk will stabilize to a limit function F_∞, that satisfies $F_\infty(\gamma(i)) = \gamma(i+1)$, $i \geq 0$, and $F_\infty(h) = h$ for any h outside of the image of γ. In particular, the identity element does not have a preimage through F_∞. Note also that the Poisson boundary of $(\mathrm{FSym}_{\mathrm{ext}}(F(X)), \mu)$ is non-trivial whenever $|X| \geq 2$.

Proposition 4.4.3. *Let H be an infinite countable group and μ a non-degenerate probability measure on $\mathrm{FSym}_{\mathrm{ext}}(H)$. Suppose that the permutation coordinate of the μ-random walk stabilizes. Then the Poisson boundary of $(\mathrm{FSym}_{\mathrm{ext}}(H), \mu)$ is non-trivial.*

Proof. Denote by $F_\infty : H \to H$ the limit function of the permutation coordinate of the random walk (F_n, S_n) on $\mathrm{FSym}_{\mathrm{ext}}(H)$. If the Poisson boundary is trivial, then the function F_∞ is the same for almost every trajectory of the random walk. Consider the left action of $\mathrm{FSym}_{\mathrm{ext}}(H)$ on the space of limit functions F_∞. The stabilization together with the non-degeneracy assumption imply that $F_\infty = f \circ F_\infty$, for all $f \in \mathrm{FSym}(H)$. This is a contradiction since every function $f \in \mathrm{FSym}(H)$ that acts non-trivially on $F_\infty(H)$ will not satisfy this equation. $\square$

A similar result holds for wreath products $A \wr B$, where it is enough to suppose that the semigroup generated by $\mathrm{supp}(\mu)$ contains two distinct elements with equal projections to B (see the proof of Theorem 3.3 in [Kaimanovich, 1991] and the discussion after Lemma 1.1 in [Erschler, 2011]). Below, we provide an example that shows that in $\mathrm{FSym}_{\mathrm{ext}}(H)$ an analogous hypothesis does not suffice to guarantee the non-triviality of the Poisson boundary.

We will consider subgroups of $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ whose projection to $\mathbb{Z}$ are proper subgroups of $\mathbb{Z}$. Among these subgroups, we can find wreath products (Proposition 4.2.4), which admit finite first moment measures with non-trivial boundary. Hence, in order to find subgroups with trivial boundary we will also need to restrict the possible values for the permutation coordinate of the elements of these subgroups.

Example 4.4.4. Let us fix $M \geq 3$ and consider the finite subset $\Sigma_M \leqslant \mathrm{FSym}(\mathbb{Z})$ formed by all $f \in \mathrm{FSym}(\mathbb{Z})$ such that $\mathrm{supp}(f) \subseteq [-M, M]$, and $f(x) = x + M$ whenever $-M \leq x \leq 0$. That is, a permutation $f \in \Sigma_M$ coincides with the identity outside of $[-M, M]$, acts as a translation by M on the interval $[-M, 0]$ and maps bijectively the set $[1, M]$ to $[-M, -1]$.

Let K be the subgroup of $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ generated by elements of the form $(f, M+1)$, for $f \in \Sigma_M$. Let us denote by S_K the set formed by the above generators, together with their inverses. Note that for $f \in \Sigma_M$, the inverse of $(f, M+1)$ is given by $(\tilde{f}, -(M+1))$, where

$$\tilde{f}(x) = \begin{cases} x + M, & \text{if } x \in [-M-1, -1], \\ -M - 1 + f^{-1}(x + M + 1), & \text{if } x \in [-2M-1, -M-2], \text{ and} \\ x, & \text{otherwise.} \end{cases}$$

In particular, we see that the values of $\tilde{f}$ are uniquely determined on the interval $[-M-1,1]$. Furthermore, when multiplying two elements $(f_1, j_1(M+1)), (f_2, j_2(M+1)) \in S_K$, one obtains

$$(f_1, j_1(M+1)) \cdot (f_2, j_2(M+1)) = (f_3, (j_1+j_2)(M+1)),$$

where

$$f_3 = f_1 \circ (j_1(M+1) \cdot f_2),$$

and the places where f_3 is not uniquely determined are a translation of those of f_2, which is a set of size M. From this, one can see that the elements of K are all of the form $g = (f, j(M+1))$, for $j \in \mathbb{Z}$ and for $f \in \mathrm{FSym}(\mathbb{Z})$ a permutation whose values are uniquely determined by the value of j, except for those of the interval $[(j-1)(M+1)+1, j(M+1)-1]$, which has size M. Further, if the element g is a product of n generators in S_K, then it holds that $|j| \leq n$. Thus, the growth function of K is bounded above by $nM!$ and as a result, the group K has a linear growth function.

Gromov's Theorem on groups of polynomial growth [Gromov, 1981b] implies that K is virtually nilpotent, and the Bass–Guivarc'h formula [Bass, 1972; Guivarch, 1973] shows that any virtually nilpotent group of linear growth is virtually $\mathbb{Z}$. We conclude that K is virtually cyclic, and hence any random walk supported on K has a trivial Poisson boundary. Nonetheless, Lemma 4.1.1 implies that the permutation coordinate will stabilize for a given measure μ with a finite first moment measure supported on K with a non-centered projection to $\mathbb{Z}$. Since we already know that the Poisson boundary is trivial, it must hold that almost every trajectory of the random walk stabilizes to the same limit function. Indeed, if the projection of μ to $\mathbb{Z}$ has positive drift, then the limit function F_∞ is almost surely given by $F_\infty(x) = x + M$, $x \in \mathbb{Z}$, whereas if the drift is negative, the limit function is $F_\infty(x) = x - M$, $x \in \mathbb{Z}$.

It is natural to draw a parallel with Proposition 4.4.3. In the above example, the permutation coordinate of every trajectory stabilizes to the same limit function F, which is not surjective. In the case of positive drift in the projection to $\mathbb{Z}$, the image of F is $\mathbb{Z} \backslash [-M, -1]$. We saw that any element $g \in K$ is of the form $g = (f, j(M+1))$, with $j \in \mathbb{Z}$ and where the values of f are uniquely determined, except for those of the interval $[(j-1)(M+1)+1, j(M+1)-1]$. When looking at the action of g on the limit function F we get

$$g \cdot F = f \circ (j(M+1) \cdot F),$$

and we note that the function $j(M+1) \cdot F$ has as its image $\mathbb{Z} \backslash [(j-1)(M+1)+1, j(M+1)-1]$, which is exactly the set where f is uniquely determined. In other words, all the elements of K act trivially on F.

We now state and prove the stabilization lemma in its general form, and then prove Lemma 4.1.1. In addition to transience of the random walk induced on H, the second hypothesis below is that $\mathbb{E}(|\mathrm{supp}(\sigma_1)|) < \infty$. This means that the number of elements on which a randomly chosen permutation σ_1 acts non-trivially has a finite expectation.

Lemma 4.4.5. *Let H be a countable group and μ be a probability measure on* $\mathrm{FSym}_{\mathrm{ext}}(H)$.

Suppose that μ induces a transient random walk on H and that $\mathbb{E}\left(|\mathrm{supp}(\sigma_1)|\right) < \infty$. Then the permutation coordinate of the μ-random walk on $\mathrm{FSym}_{\mathrm{ext}}(H)$ stabilizes.

Proof. Fix an arbitrary element $h \in H$, and for every $n \geq 1$ consider the event $A_n = \{F_{n+1}(h) \neq F_n(h)\}$. We will prove that $\sum_{n\geq 0} \mathbb{P}(A_n) < +\infty$, so that the Borel-Cantelli Lemma [Feller, 1968, Lemma VIII.3.1] will imply that almost surely only finitely many of these events happen. Since H is countable, the above implies that the stabilization of $F_n(h)$ for every $h \in H$ happens with probability 1.

Using the group operation and the definition of the action of H on $\mathrm{FSym}(H)$, we see that

$$F_{n+1}(h) = F_n \circ (S_n \cdot \sigma_{n+1})(h) = F_n\left(S_n \sigma_{n+1}(S_n^{-1}h)\right),$$

so that $A_n \subseteq \{\sigma_{n+1}(S_n^{-1}h) \neq S_n^{-1}h\}$. With this, we can obtain an upper estimate for the probability of A_n. Indeed,

$$\begin{aligned}
\mathbb{P}(A_n) &\leq \mathbb{P}\left(\sigma_{n+1}(S_n^{-1}h) \neq S_n^{-1}h\right) \\
&= \sum_{x\in H} \mathbb{P}\left(\sigma_{n+1}(x^{-1}h) \neq x^{-1}h\right) \mathbb{P}(S_n = x) \\
&= \sum_{x\in H} \mathbb{P}\left(x^{-1}h \in \mathrm{supp}(\sigma_{n+1})\right) \mathbb{P}(S_n = x) \\
&= \sum_{x\in H} \sum_{(f,y)\in \mathrm{FSym}_{\mathrm{ext}}(H)} \mathbb{1}_{\{x^{-1}h \in \mathrm{supp}(f)\}} \mu(f,y) \mathbb{P}(S_n = x).
\end{aligned}$$

Now we are going to sum over all n. Note that since we assume that the projection of μ to H is transient, there exists a constant $C > 0$ such that for every $x \in H$, $\sum_{n\geq 1} \mathbb{P}(S_n = x) < C$. Indeed, the left-hand side is the expected number of visits to x, which is equal to the probability of ever reaching x multiplied by the expected number of visits to e_H. The first term is at most 1 (since it is a probability), and the second one is a finite constant, thanks to our transience hypothesis.

We can conclude that

$$\begin{aligned}
\sum_{n\geq 1} \mathbb{P}(A_n) &\leq \sum_{n\geq 1} \sum_{x\in H} \sum_{(f,y)\in \mathrm{FSym}_{\mathrm{ext}}(H)} \mathbb{1}_{\{x^{-1}h \in \mathrm{supp}(f)\}} \mu(f,y) \mathbb{P}(S_n = x) \\
&\leq C \sum_{(f,y)\in \mathrm{FSym}_{\mathrm{ext}}(H)} \sum_{x\in H} \mathbb{1}_{\{x^{-1}h \in \mathrm{supp}(f)\}} \mu(f,y) \\
&= C \sum_{(f,y)\in \mathrm{FSym}_{\mathrm{ext}}(H)} |\mathrm{supp}(f)| \mu(f,y) = C\mathbb{E}(|\mathrm{supp}(\sigma_1)|) < \infty.
\end{aligned}$$

which is finite thanks to our hypothesis. □

Proof of Lemma 4.1.1. When H is finitely generated, so is $\mathrm{FSym}_{\mathrm{ext}}(H)$ and the condition $\mathbb{E}(|\mathrm{supp}(\sigma_1)|) < \infty$ follows from the finite first moment hypothesis. Indeed, recall the definition of S_{std} from Lemma 4.2.1. Each transposition in S_{std} changes the value of exactly two elements of H, and hence a geodesic word for (f,y) needs at least as many transpositions as half the size of the support of f. In other words, we have the inequality $|\mathrm{supp}(f)| \leq 2\|(f,y)\|_{S_{\mathrm{std}}}$ for every

$(f, y) \in \mathrm{FSym}_{\mathrm{ext}}(H)$. This implies that

$$\mathbb{E}(|\mathrm{supp}(\sigma_1)|) = \sum_{(f,y)\in \mathrm{FSym}_{\mathrm{ext}}(H)} |\mathrm{supp}(f)|\mu(f, y) \leq \sum_{(f,x)\in \mathrm{FSym}_{\mathrm{ext}}(H)} 2\|(f, x)\|_{S_{\mathrm{std}}}\mu(f, x) < \infty,$$

and so we can apply Lemma 4.4.5. □

By combining Proposition 4.4.3 with Lemma 4.4.5 we obtain the following corollary.

Corollary 4.4.6. *Let μ be a non-degenerate probability measure on $\mathrm{FSym}_{\mathrm{ext}}(H)$ that induces a transient random walk on H, such that $\mathbb{E}(|\mathrm{supp}(\sigma_1)|) < \infty$. Then the Poisson boundary of $(\mathrm{FSym}_{\mathrm{ext}}(H), \mu)$ is non-trivial.*

Remark 4.4.7. It may happen that the permutation coordinate of a random walk on $\mathrm{FSym}_{\mathrm{ext}}(H)$ with a transient projection to H does not stabilize. Indeed, whenever H is amenable, the group $\mathrm{FSym}_{\mathrm{ext}}(H)$ is amenable (Lemma 4.2.1) and it was shown by Rosenblatt [Rosenblatt, 1981] and Kaimanovich and Vershik [KaimanovichVershik, 1983, Theorem 4.4] that every amenable group admits a non-degenerate probability measure with a trivial Poisson boundary. For such μ, Proposition 4.4.3 implies that the permutation coordinate cannot stabilize. Furthermore, if H is not virtually $\mathbb{Z}$ or virtually $\mathbb{Z}^2$, the projected random walk to H will be transient, due to [Varopoulos, 1986] (see also [Woess, 2000, Theorem 3.24]).

The above remark implies that $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^d)$, $d \geq 3$, carries a random walk with a non-degenerate symmetric step distribution, with a transient projection to $\mathbb{Z}^d$ and such that the permutation coordinate does not stabilize. On the other hand, the existence of such measures for $d = 1$ or $d = 2$ is not immediate since the projected random walk to the corresponding base group could be recurrent. In the next proposition, we prove that for $H = \mathbb{Z}$ one can choose these measures so that the projection to $\mathbb{Z}$ is transient, by modifying the proofs of [KaimanovichVershik, 1983, Theorem 4.4] and [Rosenblatt, 1981, Theorem 1.10].

Proposition 4.4.8. *There exists a non-degenerate symmetric probability measure μ on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$, that induces a transient random walk on $\mathbb{Z}$ and such that the Poisson boundary of $(\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}), \mu)$ is trivial.*

Proof. Let us write $G = \mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ and $\pi : G \to \mathbb{Z}$ the projection map.

Let $\{K_i\}_{i\geq 1}$ be an increasing sequence of finite subsets of G such that $e \in K_1$ and $G = \bigcup_{i\geq 1} K_i$. Let us consider

- A sequence $\{t_i\}_{i\geq 1}$ of positive numbers such that $\sum_{i\geq 1} t_i = 1$,
- a decreasing sequence $\{\varepsilon_i\}_{i\geq 1}$ of positive numbers such that $\sum_{i\geq 1} \varepsilon_i < +\infty$, and
- sequences of integers $\{n_i\}_{i\geq 1}$ and $\{p_i\}_{i\geq 1}$ with $n_i, p_i \xrightarrow[i\to\infty]{} +\infty$ such that

$$(t_1 + t_2 + \cdots + t_{i-1})^{n_i} \leq \varepsilon_i \text{ and } (t_1 + t_2 + \cdots + t_{p_i-1})^{i} \leq \varepsilon_i.$$

For example, one can choose $t_i = 2^{-i}$, $p_i = \lfloor \log(i) \rfloor + 1$, $\varepsilon_i = \left(1 - 2^{-\log(i)}\right)^i$ and

$$n_i = \left\lceil \frac{i \log\left(1 - 2^{-\log(i)}\right)}{\log\left(1 - 2^{-i+1}\right)} \right\rceil .$$

Since G is amenable, we can find a sequence of symmetric Følner sets $\{A_m\}_{m\geq 1}$ such that, denoting $\alpha_m = \frac{1}{|A_m|}\chi_{A_m}$ the uniform probability measure on A_m, we have

$$\|\alpha_m - g\alpha_m\| \leq \varepsilon_m, \text{ for all } g \in K_m \cup (A_{m-1})^{n_m}.$$

These sets can be chosen so that A_m contains $K_m \cup (A_{m-1})^{n_m}$ and that $\pi(A_m) \subseteq \mathbb{Z}$ is a symmetric interval around 0, say $\pi(A_m) = [-N_m, N_m]$, such that $\{N_m\}_m$ are increasing positive integers, and that whenever $m \geq p_n$ we have

$$(\pi_*\alpha_m)^{*q} \left([-(n-1)N_{m-1}, (n-1)N_{m-1}]\right) < \varepsilon_n, \tag{4.2}$$

for every $1 \leq q \leq n$. In other words, we guarantee that for every $m \geq p_n$, the first n steps of the projected random walk on $\mathbb{Z}$ (whose increments distribute according to $\pi_*\alpha_m$) stay out of the set $[-(n-1)N_{m-1}, (n-1)N_{m-1}]$ with high probability. The measure we are looking for is $\mu \coloneqq \sum_{m\geq 1} t_m \alpha_m$.

Indeed, μ is a symmetric non-degenerate measure on G, which has a trivial Poisson boundary, just as in the proof [KaimanovichVershik, 1983, Theorem 4.3]. Denote $\nu = \pi_*\mu$ the projection of μ to $\mathbb{Z}$, and let us prove that ν induces a transient random walk on $\mathbb{Z}$.

By construction, we have $\nu = \sum_{m\geq 1} t_m \beta_m$, where $\beta_m = \pi_*\alpha_m$. We will prove that for every $n \geq 1$ one has $\nu^{*n}(0) \leq 2\varepsilon_n$, and since $\{\varepsilon_n\}_{n\geq 1}$ is summable, this will imply that the ν-random walk on $\mathbb{Z}$ is transient.

Note that

$$\nu^{*n} = \sum_{\mathbf{k}} t_{k_1} \cdots t_{k_n} \beta_{k_1} * \cdots * \beta_{k_n},$$

where $\mathbf{k} = (k_1, \dots, k_n)$ ranges over all possible multi-indices. We write $\nu^{*n} = \gamma_1 + \gamma_2$, where

$$\gamma_1 = \sum_{|\mathbf{k}| < p_n} t_{k_1} \cdots t_{k_n} \beta_{k_1} * \cdots * \beta_{k_n}, \text{ for } |\mathbf{k}| = \max_{1\leq i\leq n} k_i,$$

and $\gamma_2 = \nu^{*n} - \gamma_1$.

First, note that our choice of p_n guarantees that

$$\gamma_1(0) \leq \sum_{|\mathbf{k}| < p_n} t_{k_1} \cdots t_{k_n} = (t_1 + \cdots + t_{p_n - 1})^n \leq \varepsilon_n.$$

Now let us bound the value of $\gamma_2(0)$. Fix a multi-index $\mathbf{k}$ such that $|\mathbf{k}| \geq p_n$ and consider an index j such that k_j is the largest entry. Since $\mathbb{Z}$ is abelian, the convolution of measures is an abelian operation, and we can write $\beta_{k_1} * \cdots * \beta_{k_n} = \beta_{k_j}^{*q} * \theta$, where $q \geq 1$ and θ is an $(n-q)$-th convolution of β_i's with $i < k_j$, so that it satisfies

$$\operatorname{supp}(\theta) \subseteq [-(n-1)N_{k_j-1}, (n-1)N_{k_j-1}].$$

With this, we get

$$\begin{aligned}
\gamma_2(0) &= \sum_{x\in\mathbb{Z}} \beta_{k_j}^{*q}(x)\theta(-x) \\
&= \sum_{|x|\le(n-1)N_{k_j-1}} \beta_{k_j}^{*q}(x)\theta(-x) \\
&\le \sum_{|x|\le(n-1)N_{k_j-1}} \beta_{k_j}^{*q}(x) \\
&= \beta_{k_j}^{*q}\left(\left[-(n-1)N_{k_j-1}, (n-1)N_{k_j-1}\right]\right) \\
&\le \varepsilon_n,
\end{aligned}$$

where we used the fact that β_{k_j} satisfies Equation (4.2), with $k_j \ge p_n$.

We conclude that $\nu^{*n}(0) = \gamma_1(0) + \gamma_2(0) \le 2\varepsilon_n$, which finishes the proof. □

If we weaken the hypothesis $\mathbb{E}(|\operatorname{supp}(\sigma_1)|) < \infty$ from Lemma 4.4.5 it is possible that the permutation coordinate never stabilizes. Indeed, with ideas similar to an example of [Kaimanovich, 1983], we obtain the following.

Proposition 4.4.9. *The group* $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ *admits probability measures* μ *with an infinite first moment and a finite* $(1-\varepsilon)$*-moment, for every* $0 < \varepsilon < 1$*, that induce a transient random walk on* $\mathbb{Z}$ *and for which the permutation coordinate of the* μ*-random walk does not stabilize. Such measures can be chosen to satisfy* $\mathbb{E}(|\operatorname{supp}(\sigma_1)|) = \infty$ *and* $\mathbb{E}(|\operatorname{supp}(\sigma_1)|^{1-\varepsilon}) < \infty$ *for every* $0 < \varepsilon < 1$.

Proof. For each $n \ge 1$, denote by $r_n : \mathbb{Z} \to \mathbb{Z}$ the permutation

$$r_n(x) = \begin{cases} x+1, \text{ if } 0 \le x < n-1, \\ 0, \text{ if } x = n-1, \text{ and} \\ x, \text{ otherwise.} \end{cases}$$

We define the measure μ on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ as follows. Let

$$\mu((\mathsf{id}, 1)) = 1/8, \ \mu((\mathsf{id}, -1)) = 3/8,$$

and

$$\mu((r_n, 0)) = \frac{1}{2n(n+1)}, \quad \text{for } n \ge 1.$$

Note that $\sum_{n\ge1} \frac{1}{n(n+1)} = 1$, so that μ is indeed a probability measure. Also note that $|\operatorname{supp}(r_n)| = n$. From this, the fact that the harmonic series $\sum_{n\ge1} \frac{1}{n}$ diverges implies that $\mathbb{E}(|\operatorname{supp}(\sigma_1)|)$ is infinite. Moreover, since $\|(r_n, 0)\|_{S_{\mathrm{std}}} \ge |\operatorname{supp}(r_n)|$, we also have that μ has an infinite first moment. On the other hand, for every $\varepsilon > 0$ the series $\sum_{n\geq 1} \frac{n^{1-\varepsilon}}{n(n+1)}$ is convergent and thus $\mathbb{E}(|\mathrm{supp}(\sigma_1)|^{1-\varepsilon})$ is finite. The element r_n has word length at most $3n$ (since it can be expressed as the product of at most n transpositions together with $2n$ movements in the $\mathbb{Z}$ coordinate), and hence

$$\sum_{n\geq 1} \frac{\|(r_n, 0)\|_{S_{\mathrm{std}}}^{1-\varepsilon}}{n(n+1)} \leq 3^{1-\varepsilon} \sum_{n\geq 1} \frac{n^{1-\varepsilon}}{n(n+1)},$$

which is finite. Hence, μ has a finite $(1-\varepsilon)$-moment.

Let us show that the value $F_n(0)$, $n \geq 1$, almost surely changes infinitely often. By definition of the group operation and the μ-random walk, we can write $F_n = F_{n-1} \circ (S_n \cdot \sigma_n)$. Hence, $F_n(0) \neq F_{n-1}(0)$ if and only if $S_n \cdot \sigma_n(0) \neq 0$, which can be rewritten as $\sigma_n(-S_n) \neq -S_n$, by using the definition of the action of $\mathbb{Z}$ on $\mathrm{FSym}(\mathbb{Z})$ (here we use an additive notation for the group operation on $\mathbb{Z}$).

The induced random walk on $\mathbb{Z}$ is drifted to the negative numbers, and hence almost surely $S_n \xrightarrow[n\to\infty]{} -\infty$. Also, at time n the projection to $\mathbb{Z}$ satisfies $S_n \geq -n$, since the distribution of the increments of the induced random walk on $\mathbb{Z}$ is supported on $\{1, -1\}$. Consider the event $A_n = \{\sigma_n(i) \neq i \text{ for } 0 \leq i < n\}$, and note that

$$\mathbb{P}(A_n) \geq \sum_{k>n} \mathbb{P}(\sigma_n = r_k) = \sum_{k>n} \mu((r_k, 0)) = \sum_{k>n} \frac{1}{2k(k+1)} = \frac{1}{2(n+1)}.$$

Since the sequence of events $\{A_n\}_{n\geq 1}$ is independent and the series $\sum_n \mathbb{P}(A_n)$ diverges, the Borel-Cantelli Lemma implies that almost surely infinitely many of these events occur. In consequence, the value of the permutation coordinate at 0 changes infinitely often. □

In Kaimanovich's example mentioned above, the difference between the states of two adjacent lamps does stabilize [Kaimanovich, 1983], so that the Poisson boundary is non-trivial. A similar example is described by Lyons and Peres after the proof of Theorem 5.1 in [LyonsPeres, 2021a]. Examples of random walks on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$, $d \geq 1$, which have a non-trivial Poisson boundary but for which there is no functional defined by a finite set that stabilizes along infinite trajectories are given in [Erschler, 2011, Section 6].

4.5 Proof of the main theorem

Let us consider a finitely generated group H. Let μ be a probability measure on $\mathrm{FSym}_{\mathrm{ext}}(H)$ with a finite first moment and a transient projection to H. In this case, Lemma 4.1.1 implies that the permutation coordinate $(F_n)_n$ of the μ-random walk stabilizes to a limit injective function $F_\infty : H \to H$. As a result, the space $\mathcal{F}(H) := \{f : H \to H \mid f \text{ is injective}\}$ has the structure of a measure space $(\mathcal{F}(H), \lambda)$, where λ is the *hitting measure* and satisfies for any $A \subseteq \mathcal{F}(H)$ measurable, $\lambda(A) := \mathbb{P}(F_\infty \in A)$. Alternatively, λ is the push-forward of $\mathbb{P}$ through the map $\mathrm{FSym}_{\mathrm{ext}}(H)^\infty \to \mathcal{F}(H)$ that associates with every sample path of the μ-random walk on $\mathrm{FSym}_{\mathrm{ext}}(H)$ the associated limit function F_∞ of the permutation coordinate. Note that this map is shift-invariant, which implies that the measure λ is μ-stationary.

The space $(\mathcal{F}(H), \lambda)$ is thus a μ-boundary, as described in Subsection 4.3.2. That is, it is a quotient of the Poisson boundary. In this section, we prove Theorem 4.1.2, which states that for $H = \mathbb{Z}$, the μ-boundary $(\mathcal{F}(\mathbb{Z}), \lambda)$ actually coincides with the Poisson boundary of the random walk $(\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}), \mu)$.

The proof of Theorem 4.1.2 uses Kaimanovich's Conditional Entropy Criterion (Theorem 4.3.1). The main idea is that conditioned on the limit function to which the permutation coordinate converges, for every $\varepsilon > 0$ and any large enough n we can find a finite subset $Q_n \subseteq \mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ with $|Q_n| < \exp(\varepsilon n)$, and such that $(F_n, S_n) \in Q_n$ with some fixed positive probability.

The fact that μ has a finite first moment implies that the projection $\mu_{\mathbb{Z}}$ of μ to $\mathbb{Z}$ also does. Since $\mu_{\mathbb{Z}}$ induces a transient random walk, it holds that the $\mu_{\mathbb{Z}}$-random walk on $\mathbb{Z}$ has non-zero drift $\sum_{x\in\mathbb{Z}} x\mu_{\mathbb{Z}}(x)$. The law of large numbers then allows us to confine the position coordinate S_n within an interval I_n of length $2\varepsilon n$ and to estimate the values of the permutation coordinate outside this interval. However, this is not enough for our purposes, since a rough estimate for the number of values that the permutation coordinate can take inside I_n is $(2\varepsilon n)!$, which leads to sets Q_n that have an exponential size. To overcome this difficulty, we look at the possible values for the displacement of F_n. Recall that the displacement of $\sigma \in \mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ is defined as $\mathrm{Disp}(\sigma) = \sum_{i\in\mathbb{Z}} |\sigma(i) - i|$ (Definition 4.2.2). The first moment hypothesis gives us control over the possible values for the displacement associated with the permutation increments that modify the values in the interval I_n, which in turn reduces the previously mentioned estimate into a subexponential one.

4.5.1 The proof

We first state two preliminary lemmas that follow from the hypothesis of μ having a finite first moment and the Strong Law of Large Numbers [Feller, 1968, Section VIII.4]. Then, we proceed with the proof.

Lemma 4.5.1. *Consider a probability measure μ on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ with a finite first moment. Then there exists a constant $D > 0$ such that for every $\varepsilon > 0$ and almost every sequence of i.i.d. increments $\{(\sigma_k, X_k)\}_{k\geq 1}$ there exists $N \geq 1$ such that for every $n \geq N$ one has*

1. $|X_n| \leq \varepsilon n$,
2. $\sigma_n \in \mathrm{Sym}([-\varepsilon n, \varepsilon n])$, *and*
3. $\displaystyle\sum_{k=n-\varepsilon n}^{n} \mathrm{Disp}(\sigma_k) < D\varepsilon n.$

Proof. We will use the Borel-Cantelli Lemma for the first two items. We see that

$$\begin{aligned}\sum_{n\geq 1} \mathbb{P}(|X_n| \geq \varepsilon n) &= \sum_{n\geq 1} \sum_{(F,x)\in\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})} \mathbb{1}_{\{|x|\geq\varepsilon n\}} \mu(F,x) \\ &= \sum_{(F,x)\in\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})} \frac{1}{\varepsilon} |x| \mu(F,x)\end{aligned}$$

$$\leq \frac{1}{\varepsilon} \sum_{(F,x)\in \mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})} \|(F,x)\|_{S_{\mathrm{std}}} \mu(F,x) < +\infty.$$

Similarly, we have

$$\begin{aligned}\sum_{n\geq 1} \mathbb{P}\left(\mathrm{supp}(\sigma_n) \not\subseteq [-\varepsilon n, \varepsilon n]\right) &= \sum_{n\geq 1} \sum_{(F,x)\in \mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})} \mathbb{1}_{\{\mathrm{supp}(F)\not\subseteq[-\varepsilon n,\varepsilon n]\}} \mu(F,x) \\ &\leq \sum_{n\geq 1} \sum_{(F,x)\in \mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})} \mathbb{1}_{\left\{n\leq \frac{1}{\varepsilon}\max\{|y|:\ y\in \mathrm{supp}(F)\}\right\}} \mu(F,x) \\ &\leq \frac{1}{\varepsilon} \sum_{(F,x)\in \mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})} \|(F,x)\|_{S_{\mathrm{std}}} \mu(F,x) < +\infty,\end{aligned}$$

Since both sums are finite, the Borel-Cantelli Lemma implies that almost surely these events occur finitely many times, and thus neither of them happens for n sufficiently large.

The third item follows from the Strong Law of Large Numbers, since the sequence of random variables $\{\mathrm{Disp}(\sigma_k)\}_{k\geq 1}$ are i.i.d. of finite first moment. Indeed, using Lemma 4.2.3, we have

$$\mathbb{E}(\mathrm{Disp}(\sigma_1)) \leq 2\mathbb{E}(\|(\sigma_1, X_1)\|_{S_{\mathrm{std}}}) < +\infty.$$

□

The next lemma is a direct consequence of the Strong Law of Large Numbers.

Lemma 4.5.2. *Fix a probability measure ν on $\mathbb{Z}$ with finite first moment and positive drift. Denote by $\{S_n\}_{n\geq 0}$ the associated ν-random walk on $\mathbb{Z}$. Then there exists a constant $C_1 > 0$ such that for every $\varepsilon > 0$, almost surely there exists $N \geq 1$ such that for $n \geq N$ we have*

$$C_1 n - \varepsilon n \leq S_n \leq C_1 n + \varepsilon n.$$

Let us now proceed with the proof of the main theorem.

Proof of Theorem 4.1.2. Since the projection of μ to $\mathbb{Z}$ is transient and has a finite first moment, its drift $\mathbb{E}(X_1)$ is non-zero, and we lose no generality if we furthermore suppose that it is positive. We will assume this throughout the rest of the proof.

Denote by F_∞ the random variable defined as the limit of the permutation component $\{F_n\}_{n\geq 0}$ of the μ-random walk on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$, which is almost surely well-defined thanks to Lemma 4.1.1. To prove the theorem, it suffices to check the hypotheses of Theorem 4.3.1 for the boundary of limit functions F_∞.

Fix an arbitrary $\varepsilon > 0$. Thanks to Lemmas 4.5.1 and 4.5.2, there exist constants $C_1, D > 0$ and $N \geq 1$ large enough so that for every $n \geq N$ we have that

$$C_1 n - \varepsilon n \leq S_n \leq C_1 n + \varepsilon n,$$

and that the increments of the μ-random walk satisfy

$$|X_n| < \varepsilon n, \text{ and } \sigma_n \in \mathrm{FSym}([-\varepsilon n, \varepsilon n]),$$

together with

$$\sum_{k=n-\tilde{\varepsilon}n}^{n} \mathrm{Disp}(\sigma_k) < D\tilde{\varepsilon}n,$$

where $\tilde{\varepsilon} = \frac{4\varepsilon}{C_1+2\varepsilon}$. This choice of $\tilde{\varepsilon}$ is to simplify the computations below.

The above guarantees that $F_n(y) = y$ for every $y > (C_1 + 2\varepsilon)n$. Indeed, the maximum value that the projection to $\mathbb{Z}$ could have visited is $(C_1+\varepsilon)n$, and since the permutation component of the increments has a range of at most εn, there have been no modifications of the permutation component of the random walk beyond $(C_1 + \varepsilon)n + \varepsilon n = (C_1 + 2\varepsilon)n$.

Similarly, for every instant after n we know that the projection to $\mathbb{Z}$ will not visit any value smaller than $(C_1 - \varepsilon)n$, so that for every $y < (C_1 - 2\varepsilon)n$ the value $F_n(y)$ is already fixed to its limit $F_\infty(y)$. Thus, we know the exact value of $F_n(y)$ for every $y \in \mathbb{Z}$ such that $|y - C_1 n| > 2\varepsilon n$.

We now estimate the possible number of values that F_n can take in the interval $[(C_1 - 2\varepsilon)n, (C_1 + 2\varepsilon)n]$. To do so, we remark that if k_0 is the first moment that the permutation component of the random walk is not trivial on this interval, then

$$\sum_{i=(C_1-2\varepsilon)n}^{(C_1+2\varepsilon)n} |F_n(i) - i| \leq \sum_{k=k_0}^{n} \mathrm{Disp}(\sigma_k). \tag{4.3}$$

For n large enough, the smallest possible value for k_0 is $\frac{C_1-2\varepsilon}{C_1+2\varepsilon}n$. Indeed, the maximum value of S_k is $C_1 k + \varepsilon k$ and the support of the permutation component increments allows for an extra εk, so that we get the inequality

$$(C_1 + 2\varepsilon)k_0 \geq (C_1 - 2\varepsilon)n.$$

Now note that

$$n - \frac{C_1 - 2\varepsilon}{C_1 + 2\varepsilon}n = \frac{4\varepsilon}{C_1 + 2\varepsilon}n,$$

and recall that we defined $\tilde{\varepsilon} = \frac{4\varepsilon}{C_1+2\varepsilon}$, so that we have

$$\sum_{k=n-\tilde{\varepsilon}n}^{n} \mathrm{Disp}(\sigma_k) < D\tilde{\varepsilon}n = \frac{4D}{C_1 + 2\varepsilon}\varepsilon n.$$

Denote $D\prime = \frac{4D}{C_1+2\varepsilon}$. Thanks to Equation (4.3), we can interpret the above as saying that the permutation F_n must assign a non-negative value $d_i := |F_n(i) - i|$ to each element $i \in [(C_1 - 2\varepsilon)n, (C_1 + 2\varepsilon)n]$ such that

$$\sum_{i=(C_1-2\varepsilon)n}^{(C_1+2\varepsilon)n} d_i < D\prime\varepsilon n.$$

The number of ways to do this is the same as the number of ways of distributing $D\prime\varepsilon n$ identical balls into $4\varepsilon n+1$ distinguishable boxes, together with a factor of $2^{4\varepsilon n+1}$ which accounts for the fact that for the same value of d_i there are at most two choices of $F_n(i)$ (depending on whether $F_n(i) \geq i$ or $F_n(i) < i$). This gives an upper bound of

$$2^{4\varepsilon n+1} \cdot \binom{(4+D\prime)\varepsilon n}{D\prime\varepsilon n}$$

for the possible values of the function F_n.

To use Kaimanovich's criterion, we define $Q_n \subseteq \mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ to be the set of elements of the form (F, x) where $(C_1-\varepsilon)n \leq x \leq (C_1+\varepsilon)n$ and where F is a permutation as described above. The random set Q_n is measurable with respect to $\sigma(F_\infty)$. As a consequence of the above estimates, we have that $(F_n, S_n) \in Q_n$ almost surely, for n large enough, and that

$$|Q_n| \leq (4\varepsilon n+1)2^{4\varepsilon n+1} \cdot \binom{(4+D\prime)\varepsilon n}{D\prime\varepsilon n}.$$

Finally, thanks to Stirling's approximation [Feller, 1968, Section II.9], we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\log|Q_n| &\leq 4\varepsilon\log 2 + \limsup_{n\to\infty}\frac{1}{n}\log\binom{(4+D\prime)\varepsilon n}{D\prime\varepsilon n} \\ &\leq 4\varepsilon\log 2 + 4\varepsilon\log\left(\frac{4+D\prime}{4\varepsilon}\right) + D\prime\varepsilon\log\left(\frac{4+D\prime}{D\prime}\right) \\ &\leq C_2\varepsilon,\end{aligned}$$

for some constant $C_2 > 0$. With this, we have checked the hypotheses of Theorem 4.3.1 and hence finished the proof. □

4.6 Boundary triviality for recurrent projections to the base group

Theorem 4.1.2 describes the Poisson boundary of random walks on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ with a finite first moment and a transient projection to $\mathbb{Z}$. On the other hand, the situation is more delicate when the projection to $\mathbb{Z}$ is recurrent since the group $\mathrm{FSym}(\mathbb{Z})$ admits measures with a non-trivial Poisson boundary [Kaimanovich, 1983]. In this section, we prove that the Poisson boundary is trivial for finitary measures on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ or $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^2)$ that induce a recurrent random walk on the base group.

We first show the result for $H=\mathbb{Z}$, following similar ideas to [KaimanovichVershik, 1983, Proposition 6.2].

Proposition 4.6.1. *Consider μ a finitely supported probability measure on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ such that its projection to $\mathbb{Z}$ induces a recurrent random walk. Then the Poisson boundary of $(\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}), \mu)$ is trivial.*

Proof. Denote by ν the projection of μ to $\mathbb{Z}$, which has a zero mean and is finitely supported.

Consider the μ-random walk $\{(F_n, S_n)\}_{n\geq 0}$ on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$. We first recall that Kolmogorov's Maximal Inequality [Feller, 1968, Section IX.7] states that for any $\lambda > 0$, we have

$$\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| \geq \lambda\right) \leq n\sigma^2/\lambda^2,$$

where σ^2 is the variance of ν. In particular for $\lambda = n^{3/4}$, we get

$$\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| < n^{3/4}\right) \geq 1 - \frac{\sigma^2}{n^{1/2}}.$$

Thus, with probability at least $1 - \sigma^2/n^{1/2}$, the random walk at time n belongs to the set

$$A_n = \left\{(f, x) \in G \mid |x| \leq n^{3/4},\ \mathrm{supp}(f) \subseteq [-n^{3/4} - M, n^{3/4} + M]\right\},$$

where $M = \max\{C \geq 0 \mid \text{for every } (f, x) \in \mathrm{supp}(\mu) \text{ and } |y| > C,\ f(y) = y\}$ is the size of the largest support of functions f that participate in the support of μ.

The size of the set A_n is subexponential. Indeed, we have

$$\frac{1}{n}\log|A_n| \leq \frac{1}{n}\log\left((2n^{3/4} + 1)\cdot(2n^{3/4} + 1 + M)!\right),$$

which converges to 0. This follows by applying Stirling's approximation to bound from above the term $\log((2n^{3/4} + 1 + M)!)$. We can thus apply Theorem 4.3.1 with the trivial boundary since the sets A_n are deterministic and we have

$$\mathbb{P}\left((F_n, S_n) \in A_n\right) \geq \mathbb{P}\left(\max_{1\leq k\leq n} |S_k| < n^{3/4}\right) \geq 1 - \frac{\sigma^2}{n^{1/2}} > 1/2,$$

for any n large enough. □

In order to prove the analogous result for $H = \mathbb{Z}^2$, we need more detailed calculations. Indeed, in the proof of Proposition 4.6.1 we argued that the projection of the random walk to $\mathbb{Z}$ at time n stayed inside an interval of length of order $n^{3/4}$ with some positive probability. Then a rough estimate for the total number of permutations in this interval gives rise to a subset of $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z})$ whose size is of order $(n^{3/4})!$. This grows subexponentially and hence we can apply Theorem 4.3.1. In the case of $\mathbb{Z}^2$, we can similarly argue that with a fixed positive probability, the projection of the random walk to $\mathbb{Z}^2$ at time n has not visited the outside of a ball of radius $C_1 n^{1/2}$, for some constant $C_1 > 0$. However, since the base group now has quadratic growth, a rough estimate for the number of permutations gives an exponential number of possibilities, and so we cannot apply Theorem 4.3.1. We go past these complications by giving more detailed estimates for the range of the projected random walk to $\mathbb{Z}^2$, which is $O(n/\log(n))$, and by looking at the displacement of the permutation coordinate increments.

Proposition 4.6.2. *Consider μ a finitely supported probability measure on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^2)$ such that its projection to $\mathbb{Z}^2$ induces a recurrent random walk. Then the Poisson boundary of $(\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^2), \mu)$ is trivial.*

Proof. By using Kolmogorov's maximal inequality on both coordinates of $\mathbb{Z}^2$, we see that for a large enough constant $C_1 > 0$ we have

$$\mathbb{P}\left(\max_{1\le k\le n} \|S_k\| \le C_1 n^{1/2}\right) > 1/2$$

for all $n \ge 1$, where $\|\cdot\|$ denotes the L_1 norm on $\mathbb{Z}^2$. That is, $\|(x_1,x_2)\| = |x_1| + |x_2|$, for $(x_1,x_2) \in \mathbb{Z}^2$.

We need to estimate the n-th instant (F_n, S_n) of the random walk. Let us denote by $R_n \coloneqq |\{S_0, S_1, \dots, S_n\}|$ the range of the induced random walk on $\mathbb{Z}^2$. In other words, R_n is the number of distinct elements of $\mathbb{Z}^2$ visited up to time n. Since we assume that μ has finite support, the following limit

$$\lim_{n\to\infty} \frac{R_n}{n/\log(n)}$$

exists almost surely and is positive. This was first proved for the simple random walk on $\mathbb{Z}^2$ in [DvoretzkyErdös, 1951]. The same arguments in that paper prove that for any finitely supported μ such that the induced random walk on $\mathbb{Z}^2$ is recurrent, one has $\mathbb{E}(R_n) = C\frac{n}{\log(n)} + o\left(\frac{n}{\log(n)}\right)$ for a constant $C > 0$. In addition, it is proved in [JainPruitt, 1972] that for μ as above, the strong law of large numbers $\lim_{n\to\infty}\frac{R_n}{E(R_n)} = 1$ holds almost surely. The combination of both results implies the statement above. In particular for our proof, the above implies that there exists a constant $C_2 > 0$ such that almost surely for every large enough n we have $R_n < C_2\frac{n}{log(n)}$.

With this, the n-th instant of the random walk is determined by choosing at most $C_2\frac{n}{\log(n)}$ elements of $\mathbb{Z}^2$ to visit inside the ball of radius $C_1\sqrt{n}$, then choosing one of these elements to be the final position S_n of the random walk on $\mathbb{Z}^2$, and finally choosing the values for the permutation coordinate F_n. The first two terms above give a factor of

$$\binom{(C_1\sqrt{n}+1)^2}{C_2\frac{n}{\log(n)}} \cdot C_2\frac{n}{\log(n)},$$

which grows subexponentially. Indeed, we note that both $\frac{1}{n}\log(n/\log(n)) = \frac{1}{n}\log(n) - \frac{1}{n}\log\log n$, and

$$\frac{1}{n}\log\binom{(C_1\sqrt{n}+1)^2}{C_2\frac{n}{\log(n)}}$$

converge to 0, as it follows from Stirling's approximation.

In order to estimate the number of possible values for the permutation coordinate F_n, we look at the displacement function. Note that we have $\sum_{x\in\mathbb{Z}^2}\|x - F_n(x)\| \le C_3 n$, for some constant $C_3 > 0$. Indeed, since μ has finite support, each increment will map each element to its image in a uniformly bounded neighborhood, and so the total displacement at every step is bounded by a fixed constant.

With the above, we can associate to the permutation $F_n(x)$ values $d_x \ge 0$, for each x in the support of size $C_2\frac{n}{\log(n)}$, whose sum must be at most $C_3 n$. The total number of ways of assigning these numbers to a fixed support is given by

$$\binom{C_3 n + C_2 \frac{n}{\log(n)} - 1}{C_3 n}.$$

Suppose that we have fixed a support of size $C_2 \frac{n}{\log(n)}$ as well as the displacements of each element $\{d_x\}_x$. We remark that for every element x, its image $F_n(x)$ can be any element that satisfies $\|x - F_n(x)\| = d_x$. For a fixed value of d_x there are $2d_x + 1$ such elements, and hence the total number of permutations for this fixed support and displacement is bounded above by $\prod_{i=1}^{C_2 \frac{n}{\log(n)}} (2d_i + 1)$. Since we have the constraint $\sum_{i=1}^{C_2 \frac{n}{\log(n)}} d_i = C_3 n$, the above product is maximized when all the values $2d_i + 1$ are equal, and hence we have the upper bound

$$\left(\frac{2C_3 n + C_2 \frac{n}{\log(n)}}{C_2 \frac{n}{\log(n)}} \right)^{C_2 \frac{n}{\log(n)}}.$$

This also grows subexponentially.

In conclusion, our analysis shows that with a probability of at least $1/2$, the n-th instant (F_n, S_n) of the random walk on $\mathrm{FSym}_{\mathrm{ext}}(\mathbb{Z}^2)$ belongs to a set A_n of size bounded above by

$$\binom{(C_1\sqrt{n}+1)^2}{\frac{C_2 n}{\log(n)}} \cdot \frac{C_2 n}{\log(n)} \cdot \binom{C_3 n + \frac{C_2 n}{\log(n)} - 1}{C_3 n} \cdot \left(\frac{2C_3 n + \frac{C_2 n}{\log(n)}}{\frac{C_2 n}{\log(n)}} \right)^{\frac{C_2 n}{\log(n)}},$$

which grows subexponentially. We can thus apply Theorem 4.3.1 to conclude that the Poisson boundary is trivial. $\square$

Chapter 5

The Poisson boundary of wreath products

This chapter corresponds to the preprint [FrischSilva, 2023].

Abstract

We give a complete description of the Poisson boundary of wreath products $A \wr B := \bigoplus_B A \rtimes B$ of countable groups A and B, for probability measures μ with finite entropy where lamp configurations stabilize almost surely. If, in addition, the projection of μ to B is Liouville, we prove that the Poisson boundary of $(A \wr B, \mu)$ is equal to the space of limit lamp configurations, endowed with the corresponding hitting measure. In particular, this answers an open question asked by Kaimanovich [Kaimanovich, 2001] and Lyons-Peres [LyonsPeres, 2021a] for $B = \mathbb{Z}^d$, $d \geq 3$, and measures μ with a finite first moment.

This is joint work with Joshua Frisch.

Contents

5.1 Introduction

The classical *Poisson integral representation formula* (see e.g. [Ahlfors, 1978, Theorems 4.22 and 4.23]) establishes an isometric isomorphism between the Banach space of bounded harmonic functions on the disk $\mathbb{D}$ and the space of bounded measurable functions on the circle $\partial\mathbb{D}$. It is possible to find an analogous duality for countable groups as follows. Let G be a countable group and μ a probability measure on G. A function $f : G \to \mathbb{R}$ is said to be μ-harmonic if $f(g) = \sum_{h \in G} f(gh)\mu(h)$ for every $g \in G$. The *Poisson boundary* of (G, μ) is a measure space endowed with a measurable G-action with the following property: there is an isometric isomorphism between the space of bounded μ-harmonic functions on G and the space of bounded measurable functions on the Poisson boundary (see [Furstenberg, 1971, Theorem 3.1] and the introductions of [KaimanovichVershik, 1983; Kaimanovich, 2000]). Equivalently, the Poisson boundary encodes the asymptotic behavior of the trajectories of the *μ-random walk* $\{w_n\}_{n\geq 0}$ on G, which is the Markov chain with state space G and with independent increments $\{w_{n-1}^{-1}w_n\}_{n\geq 1}$ distributed according to μ.

Let us denote by $(G^{\mathbb{N}}, \mathbb{P})$ the space of sample paths of the μ-random walk on G.

Definition 5.1.1. Let G be a countable group, and let μ be a probability measure on G. Two sample paths $\mathbf{w} = (w_1, w_2, \ldots)$, $\mathbf{w}' = (w_1', w_2', \ldots) \in G^{\mathbb{N}}$ are said to be equivalent if there exist $p, N \geq 0$ such that $w_n = w_{n+p}'$ for all $n > N$. Consider the measurable hull associated with this equivalence relation. That is, the σ-algebra formed by all measurable subsets of the space of trajectories $(G^{\mathbb{N}}, \mathbb{P})$ which are unions of the equivalence classes of $\sim$ up to $\mathbb{P}$-null sets. The associated quotient space is called the *Poisson boundary* of the random walk (G, μ).

There are several other equivalent definitions of the Poisson boundary, and we mention some of them in Section 5.2.

The measure μ is called *adapted* (resp. *non-degenerate*) if the support of μ generates G as a group (resp. as a semigroup). Another name for adapted measures in the literature is *irreducible*. We say that an adapted probability measure μ on a group G has the *Liouville property* if every bounded μ-harmonic function is constant. This is equivalent to (G, μ) having a trivial Poisson boundary. A natural question to ask is whether a random walk (G, μ) has the Liouville property. If the random walk does not have this property, the next problem is to realize the Poisson boundary as an explicit measure space related to the geometry of G. These problems have been studied in the last decades for various families of groups under different conditions on the measure μ. We refer to Subsection 5.1.2 for a more detailed description of some of these results and the corresponding references.

A key quantity in the study of Poisson boundaries is *entropy*. Given a probability measure μ on G, its entropy is $H(\mu) := -\sum_{g\in G} \mu(g)\log(\mu(g))$.

Definition 5.1.2. The entropy of the random walk (G, μ), also called the *asymptotic entropy*, is defined as

$$h_\mu := \lim_{n\to\infty} \frac{H(\mu^{*n})}{n}.$$

This quantity was introduced by Avez [Avez, 1972], who proved that if μ is finitely supported and $h_\mu = 0$ then (G, μ) has a trivial Poisson boundary. Furthermore, for any probability measure with $H(\mu) < \infty$, the *entropy criterion* of Derriennic [Derriennic, 1980] and Kaimanovich-Vershik [KaimanovichVershik, 1983] states that the triviality of the Poisson boundary is equivalent to the vanishing of h_μ. This criterion was extended to a *conditional entropy criterion* by Kaimanovich [Kaimanovich, 1985, Theorem 2], [Kaimanovich, 2000, Theorem 4.6], that establishes whether a candidate for the Poisson boundary is equal to it. This result is often applied via the associated *ray criterion* [Kaimanovich, 2000, Theorem 5.5] and *strip criterion* [Kaimanovich, 2000, Theorem 6.4], which are deduced from it.

It is known that any adapted random walk on an abelian group has a trivial Poisson boundary [Blackwell, 1955] (see also [ChoquetDeny, 1960; DoobSnellWilliamson, 1960]). The same holds for nilpotent groups, which was first proved for non-degenerate probability measures in [DynkinMaljutov, 1961] and for general probability measures in [Azencott, 1970, Théorème IV.1] (see also [Jaworski, 2004, Lemma 4.7]). For non-degenerate measures, it is furthermore proved in [Margulis, 1966] that every positive harmonic function on a nilpotent group is constant. Generalizing the case of nilpotent groups, the Poisson boundary is trivial for any adapted measure on a countable hyper-FC-central group [Jaworski, 2004; LinZaidenberg, 1998], [ErschlerKaimanovich, 2023, Proposition 5.10]. These results are satisfied by all adapted step distributions without requiring any additional hypothesis. In contrast, results describing a *non-trivial* Poisson boundary of a random walk on a group usually assume some control over the tail decay of the measure μ, commonly through the finiteness of some moment. The only known results that describe a non-trivial Poisson boundary for all probability measures μ with finite entropy, without assuming extra conditions on μ, are [ForghaniTiozzo, 2019, Theorem 1.2] for free non-abelian semigroups and [ChawlaForghaniFrischTiozzo, 2022] for hyperbolic groups and more generally acylindrically hyperbolic groups.

There are families of groups that possess a geometric boundary that, for any adapted random walk, can be endowed with a probability measure so that the associated probability space is an equivariant quotient of the Poisson boundary. Such a quotient is called a *μ-boundary*, see Subsection 5.2.2. This is the case for any adapted random walk on a free group F_k, for which the geometric boundary is the space ∂F_k of infinite reduced words [Furstenberg, 1967, Section 4], and more generally, for hyperbolic groups with the Gromov boundary (see [Woess, 1993, Corollary 1] and [Kaimanovich, 2000, Theorem 7.6]). Hence, in these cases it is a natural problem to ask whether the geometric boundary corresponds to the Poisson boundary, without any extra assumptions on the measure μ. This is currently an open problem, and there are no results in this direction for adapted probability measures with infinite entropy. The situation is different for amenable groups. Amenability is equivalent to the existence of an adapted probability measure with a trivial Poisson boundary, and thus it is not possible to find a geometric boundary that describes the Poisson boundary for all adapted measures. An important family of groups covered by the results of the current paper are wreath products $A \wr \mathbb{Z}^d$, $d \geq 1$, for A a countable group. If A is amenable, then $A \wr \mathbb{Z}^d$ is also amenable. Our results provide a complete description of the Poisson boundary of random walks on $A \wr \mathbb{Z}^d$, whenever the step distribution of the random walk has finite entropy and where the natural candidate for the Poisson boundary of a wreath product is well-defined (see Theorem 5.1.3 and Corollary 5.1.4 below).

We also remark that there are almost no results that identify a non-trivial Poisson boundary for infinite entropy measures on groups. One exception that we are aware of is [ErschlerKaimanovich, 2023, Theorem A], which describes the Poisson boundary of countable ICC groups for measures from the construction of [FrischHartmanTamuzVahidi Ferdowsi, 2019], that can be chosen to have infinite entropy. We mention that [ForghaniKaimanovich, 2015] describes the Poisson boundary of the free semigroup for measures with a finite first logarithmic moment without assuming finite entropy (see also [ForghaniTiozzo, 2019, Theorem 1.2] and [Forghani, 2015, Theorem 3.6.3]).

5.1.1 Main results

Given A and B countable groups, their *wreath product* $A \wr B$ is the semi-direct product $\bigoplus_B A \rtimes B$, where B acts on the direct sum by translations (we remark that some authors use the notation $B \wr A$). Any random walk $\{w_n\}_{n\geq 0}$ on $A \wr B$ can be decomposed using the semi-direct product structure as $w_n = (\varphi_n, X_n)$, $n \geq 0$. Here $\varphi_n \in \bigoplus_B A$ is called the *lamp configuration at instant* n, and in general it is not a random walk on $\bigoplus_B A$. On the other hand, X_n is called the *base position at instant* n, and it does describe a random walk on the *base group* B.

Kaimanovich and Vershik [KaimanovichVershik, 1983, Section 6] studied the groups $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$, $d \geq 1$, and showed that if the step distribution is finitely supported and the random walk $\{X_n\}_{n\geq 0}$ on $\mathbb{Z}^d$ is transient, then the lamp configurations $\{\varphi_n\}_{n\geq 0}$ almost surely stabilize to a *limit lamp configuration* φ_∞ (see Definition 5.3.1). If the random walk is non-degenerate, this implies the non-triviality of the associated Poisson boundary [KaimanovichVershik, 1983, Propositions 6.1 and 6.4]. In contrast, if the induced random walk to $\mathbb{Z}^d$ is recurrent, the Poisson boundary is trivial [KaimanovichVershik, 1983, Proposition 6.3]. Kaimanovich and Vershik asked whether, in the transient case and for finitely supported probability measures, the convergence to the limit lamp configuration completely describes the Poisson boundary. It is natural to extend the formulation of this question to arbitrary probability measure for which lamp configurations stabilize almost surely along sample paths of the μ-random walk.

Our first main result provides a description of the Poisson boundary of $A \wr B$, under general conditions on the measure μ.

Theorem 5.1.3. *Consider non-trivial countable groups A and B. Let μ be a probability measure on $A \wr B$ with finite entropy and such that lamp configurations stabilize almost surely. Denote by $(\partial B, \nu_B)$ the Poisson boundary of the induced random walk on B. Then the Poisson boundary of $(A \wr B, \mu)$ is completely described by the space $A^B \times \partial B$, endowed with the corresponding hitting measure.*

If, in addition, the induced random walk on B has a trivial Poisson boundary (in particular whenever $B = \mathbb{Z}^d$), Theorem 5.1.3 implies the following.

Corollary 5.1.4. *Consider non-trivial countable groups A and B. Let μ be a probability measure on $A \wr B$ with finite entropy and such that lamp configurations stabilize almost surely. Suppose that the Poisson boundary of the induced random walk on B is trivial. Then the space of infinite lamp configurations A^B endowed with the hitting measure is the Poisson boundary of $(A \wr B, \mu)$.*

The Poisson boundary of wreath products has been previously described under various conditions on the groups A and B, as well as on the measure μ [Erschler, 2011; JamesPeres, 1996; Kaimanovich, 2001; KarlssonWoess, 2007; LyonsPeres, 2021a; Sava, 2010]; we explain these results in Subsection 5.1.2. These descriptions of the Poisson boundary use the space of limit lamp configurations, and the conditions on μ ensure that the stabilization phenomenon does occur. In particular, a sufficient condition for the stabilization of lamp configurations is that the measure

μ has a finite first moment and that the induced random walk on B is transient [Erschler, 2011, Lemma 1.1] (see also [Kaimanovich, 1991, Theorem 3.3] for $B = \mathbb{Z}^d$, and [KarlssonWoess, 2007, Theorem 2.9] for B a free group). For the original groups studied by Kaimanovich and Vershik, Erschler proved that the Poisson boundary of a non-degenerate random walk on $A \wr \mathbb{Z}^d$, with $d \geq 5$, is equal to the space of limit lamp configurations, with the hypothesis that A is finitely generated and that μ has a finite third moment [Erschler, 2011, Theorem 1]. Later, Lyons and Peres proved that this also holds for $d = 3, 4$, improving the assumption on μ by requiring only a finite second moment [LyonsPeres, 2021a, Theorem 5.1].

Theorem 5.1.3 generalizes all previous descriptions of the Poisson boundary of wreath products to a setting where there is no control over the tail decay of the measure μ and the previous techniques do not work. For our result to hold, we must add the hypothesis of stabilization of lamp configurations along sample paths. This is not always the case, and there are examples where this does not happen for measures μ with a finite $(1-\varepsilon)$-moment, for any $\varepsilon > 0$ [Kaimanovich, 1983, Proposition 1.1] (see also [Erschler, 2011, Section 6] and the last paragraph of Section 5 in [LyonsPeres, 2021a]). Nonetheless, it is true that any non-degenerate random walk on $A \wr B$ with finite entropy and a transient projection to B has a non-trivial Poisson boundary [Erschler, 2004b, Theorem 3.1]. The description of the Poisson boundary seems more challenging in the case where there is no stabilization of the lamp configurations, as there are not even any known candidates for the Poisson boundary.

The following is a consequence of Theorem 5.1.3.

Corollary 5.1.5. *Let A be a non-trivial finitely generated group, and let $d \geq 3$. Let μ be a non-degenerate probability measure on $A \wr \mathbb{Z}^d$ with a finite first moment. Then the Poisson boundary of $(A \wr \mathbb{Z}^d, \mu)$ is equal to the space of infinite lamp configurations endowed with the corresponding hitting measure.*

This was an open question, asked by Kaimanovich [Kaimanovich, 2001, Example 3.6.7] and by Lyons-Peres at the end of Section 5 in [LyonsPeres, 2021a]. We remark that [Kaimanovich, 2001, Theorem 3.6.6] partially answers this question for probability measures where the projection to $\mathbb{Z}^d$ has non-zero mean. This covers in particular all probability measures on $A \wr \mathbb{Z}$ with a finite first moment and a transient projection to $\mathbb{Z}$, and all probability measures on $A \wr \mathbb{Z}^2$ with a finite second moment and a transient projection to $\mathbb{Z}^2$. Indeed, the transience of the induced random walk on $\mathbb{Z}$ (resp. $\mathbb{Z}^2$) with a finite first moment (resp. finite second moment) hypothesis implies that the induced random walk on $\mathbb{Z}$ (resp. $\mathbb{Z}^d$) has non-zero mean [Spitzer, 1976, Theorem 8.1].

Our second main theorem is an extension of Corollary 5.1.5 to a more general setting, where the induced random walk on the base group B does not necessarily have the Liouville property. Below, we denote by $\langle \mathrm{supp}(\mu) \rangle_+$ the semigroup generated by the support of μ.

Theorem 5.1.6. *Let A, B be finitely generated groups and let μ be a probability measure on $A \wr B$ such that $\langle \mathrm{supp}(\mu) \rangle_+$ contains two distinct elements with the same projection to B. Assume that μ has a finite first moment and that it induces a transient random walk on B. Then the space* *of infinite lamp configurations A^B endowed with the corresponding hitting measure is equal to the Poisson boundary of $(A \wr B, \mu)$.*

The proofs of Theorems 5.1.3 and 5.1.6 use Kaimanovich's conditional entropy criterion [Kaimanovich, 1985, Theorem 2], [Kaimanovich, 2000, Theorem 4.6]. We refer to Subsection 5.1.3 for a sketch of our argument.

The key idea behind the proof of Theorem 5.1.6 is that the extra hypothesis of finite first moment allows us to recover the Poisson boundary of the projected random walk on B via the limit lamp configuration (Proposition 5.5.2). Note that the assumption that $\langle \mathrm{supp}(\mu)\rangle_+$ contains two distinct elements with the same projection to B is necessary in order to do this, since otherwise the measure μ could be supported in a conjugate of the base group B and then the limit lamp configuration does not encode information about the trajectory of the random walk in the base group (see Example 5.5.5).

It remains an open question whether one can recover the information about the Poisson boundary in the base group from the limit lamp configuration without the assumption of a finite first moment, which would improve Theorem 5.1.3.

Question 5.1.7. Consider the hypotheses of Theorem 5.1.3, and suppose that the projected random walk to B has a non-trivial Poisson boundary. Is the Poisson boundary of $(A \wr B, \mu)$ the space of limit lamp configurations A^B endowed with the corresponding hitting measure?

We finish this subsection with an application of our main theorems to groups other than wreath products. Indeed, Theorem 5.1.3 does not have a non-degeneracy assumption, and Theorem 5.1.6 holds for a large class of possibly degenerate probability measures with a finite first moment, so that both of these results can be applied for random walks supported on proper subgroups of wreath products. In particular, this allows us to identify the Poisson boundary of random walks on free solvable groups, with adequate assumptions on the step distribution of the random walk.

Let F_d be a free group of rank $d \geq 1$, consider $N \lhd F_d$ a normal subgroup and let us denote by $[N, N]$ the subgroup of F_d generated by all commutators $[n_1, n_2] := n_1 n_2 n_1^{-1} n_2^{-1}$ of arbitrary elements $n_1, n_2 \in N$. In what follows we consider groups of the form $F_d/[N, N]$. Let us introduce the notation $F_d^{(0)} := F_d$ and $F_d^{(k)} := \left[F_d^{(k-1)}, F_d^{(k-1)}\right]$ for $k \geq 1$. Then, choosing $N = F_d^{(k)}$ in the above construction gives rise to the group $S_{d,k} := F_d/F_d^{(k)}$, which is called the *free solvable group of rank d and class k*. Note that every solvable group of class k generated by d elements is a quotient of the group $S_{d,k}$. For $k = 1$ one obtains $S_{d,1} = \mathbb{Z}^d$ the free abelian group of rank d, and in the case $k = 2$ the group $S_{d,2}$ is commonly referred to as the *free metabelian group of rank d*.

The well-known *Magnus embedding* expresses the group $F/[N, N]$ as a subgroup of the wreath product $\mathbb{Z}^d \wr F/N$. This result was proved by Magnus [Magnus, 1939], who described an explicit embedding of the group $F/[N, N]$ into a matrix group with coefficients in a module over the group ring $\mathbb{Z}(F_d/N)$, which is isomorphic to $\mathbb{Z}^d \wr F/N$. This embedding together with its description via Fox's free differential calculus [ChenFoxLyndon, 1958; Fox, 1953; Fox, 1954; Fox, 1956; Fox, 1960] have been extensively used in the past decades to study algorithmic and geometric properties of free solvable groups; see e.g. [DromsLewinServatius, 1993; MyasnikovRomankovUshakovVershik, 2010; RemeslennikovSokolov, 1970; Sale, 2015; Ushakov, 2014; Vassileva, 2011; Vassileva, 2012; Vershik, 2000b].

Let $X \subseteq F_d/N$ be the image of a free generating set of F_d via the canonical quotient epimorphism and let us consider the labeled, directed, Cayley graph $\mathrm{Cay}(F_d/N)$ of the group F_d/N with respect to X. Denote by $\mathrm{Edges}(F_d/N)$ the set of labeled and directed edges of $\mathrm{Cay}(F_d/N)$. As explained in [DromsLewinServatius, 1993; MyasnikovRomankovUshakovVershik, 2010; Saloff-CosteZheng, 2015; Vershik, 2000b], the Magnus embedding together with a geometric interpretation of Fox derivatives gives a way of associating with every group element $g \in F_d/[N,N]$ a unique finitely supported *flow* $\mathfrak{f}_g : \mathrm{Edges}(F_d/N) \to \mathbb{Z}$. We recall this description in more detail in Subsections 5.6.1 and 5.6.2. In the context of random walks on groups, this interpretation has been used for the description of the Poisson boundary in the case of free metabelian groups [Erschler, 2011; LyonsPeres, 2021a], and for the computation of the asymptotics of the return probability of symmetric finitely supported probability measures on free solvable groups in [Saloff-CosteZheng, 2015]. In particular, it is proved in [Erschler, 2011, Lemma 1.2] that if μ is a probability measure on $F/[N,N]$ with a finite first moment and that induces a transient random walk on F/N, then for almost every sample path $\{w_n\}_{n\geq 0}$ of the μ-random walk on $F/[N,N]$, the associated flows $\{\mathfrak{f}_{w_n}\}_{n\geq 0}$ stabilize to a limit function $\mathfrak{f}_\infty : \mathrm{Edges}(F_d/N) \to \mathbb{Z}$. Hence, under the previous hypotheses on μ, the space $\mathbb{Z}^{\mathrm{Edges}(F_d/N)}$ endowed with the corresponding hitting measure is a μ-boundary for $F_d/[N,N]$. The space $\mathbb{Z}^{\mathrm{Edges}(\mathbb{Z}^d)}$ is known to be the Poisson boundary of the free metabelian group $S_{d,2}$ of rank $d \geq 5$ for adapted probability measures μ with a finite third moment [Erschler, 2011, Theorem 2], and for $d \geq 3$ and μ with a finite second moment [LyonsPeres, 2021a, Section 6]. The following is the principal result that we obtain in this context.

Corollary 5.1.8. *Let N be a normal subgroup of a free group F_d, and consider a probability measure μ on $F_d/[N,N]$.*

1. *Suppose that $H(\mu) < \infty$ and that the flows associated with the μ-random walk on $F_d/[N,N]$ stabilize almost surely. Denote by $(\partial(F_d/N), \nu_{F_d/N})$ the Poisson boundary of the random walk induced by μ on F_d/N. Then the Poisson boundary of $(F_d/[N,N], \mu)$ is the space $\mathbb{Z}^{\mathrm{Edges}(F_d/N)} \times \partial(F_d/N)$, endowed with the corresponding hitting measure.*
2. *Suppose that μ has a finite first moment. Assume furthermore that $\langle \mathrm{supp}(\mu)\rangle_+$ contains two distinct elements with the same projection to F_d/N, and that μ induces a transient random walk on F_d/N. Then the Poisson boundary of $(F_d/[N,N], \mu)$ is the space $\mathbb{Z}^{\mathrm{Edges}(F_d/N)}$, endowed with the corresponding hitting measure.*

When $N = F_d^{(k)}$ is the k-th derived subgroup of F_d, Theorem 5.1.8 states that for any probability measure μ on the free solvable group $S_{d,k}$ such that $H(\mu) < \infty$ and for which flows stabilize almost surely along sample paths of the μ-random walk, the Poisson boundary of $(S_{d,k}, \mu)$ is the space $\mathbb{Z}^{\mathrm{Edges}(S_{d,k-1})} \times \partial(S_{d,k-1})$ endowed with the hitting measure. In particular, if μ has a finite first moment then $H(\mu) < \infty$, and thanks to [Erschler, 2011, Lemma 1.2] the flows associated with the μ-random walk stabilize almost surely, so that Theorem 5.1.8 applies.

With this, we obtain the description of the Poisson boundary for all random walks on any free solvable group with a step distribution with a finite first moment.

Corollary 5.1.9. *Let $S_{d,k}$ be the free solvable group of rank $d \geq 2$ and derived length $k \geq 2$, and let μ be an adapted probability measure on $S_{d,k}$ with a finite first moment.*

1. *If $d = k = 2$, then the Poisson boundary of $(S_{2,2}, \mu)$ is non-trivial if and only if the projection of μ to $\mathbb{Z}^2$ induces a transient random walk In this case, the Poisson boundary is the space $\mathbb{Z}^{\mathrm{Edges}(\mathbb{Z}^2)}$, endowed with the corresponding hitting measure.*
2. *If $\max\{d, k\} \geq 3$, then the Poisson boundary of $(S_{d,k}, \mu)$ is non-trivial and it is the space $\mathbb{Z}^{\mathrm{Edges}(S_{d,k-1})}$, endowed with the corresponding hitting measure.*

This corollary generalizes the results of [Erschler, 2011, Theorem 2] for free metabelian groups $S_{d,2}$ with $d \geq 5$ and μ with a finite third moment, and [LyonsPeres, 2021a, Section 6] for $S_{d,k}$, with $d \geq 3$ and μ with a finite second moment.

5.1.2 Background

The interest in considering in Theorem 5.1.3 a class of infinitely supported probability measures with no extra moment assumptions, in particular with an infinite first moment, comes from the fact that such measures appear naturally in the study of Poisson boundaries. One such occurrence is in the study of groups of intermediate growth. Random walks on groups of intermediate growth with a non-trivial Poisson boundary and a known control of the tail decay of the step distribution have been constructed in [Erschler, 2004a, Theorem 2], and more recently in [ErschlerZheng, 2020, Theorems A, B and C] where in particular near-optimal lower bounds for the volume growth of Grigorchuk's group are obtained. Such measures must necessarily have an infinite first moment, since the entropy criterion shows that any probability measure with a finite first moment in a group of subexponential growth has a trivial Poisson boundary. We refer to [ErschlerZheng, 2020, Section 2.1] and [Zheng, 2023, Section 2] for a more detailed explanation of the relation between random walks with a non-trivial Poisson boundary and estimates for the growth of a group. Additionally, we mention that groups of exponential growth for which every finitely supported measure has a trivial Poisson boundary are constructed in [BartholdiErschler, 2017, Theorem A]. It is currently an open problem whether there exist groups of exponential growth for which every probability measure with a finite first moment has a trivial Poisson boundary [BartholdiErschler, 2017, Question 1.1].

Another setting where infinitely supported measures appear naturally is in the study of amenable groups. On the one hand, the amenability of a group is entirely determined by the existence of random walks with non-trivial boundaries. More precisely, every non-degenerate random walk in a non-amenable group has a non-trivial Poisson boundary [Azencott, 1970, Proposition II.1] (see also the last two paragraphs of Section 9 in [Furstenberg, 1973]), and every amenable group G admits a non-degenerate symmetric measure μ with $\mathrm{supp}(\mu) = G$ and with a trivial Poisson boundary [Rosenblatt, 1981, Theorem 1.10], [KaimanovichVershik, 1983, Theorem 4.4]. On the other hand, the class of amenable groups that admit non-degenerate measures with a non-trivial Poisson boundary coincides with the class of amenable groups that are not hyper-FC-central [FrischHartmanTamuzVahidi Ferdowsi, 2019, Theorem 1] (the fact that any random walk on a hyper-FC central countable group has a trivial Poisson boundary had previously been proved in [Jaworski, 2004; LinZaidenberg, 1998]). We recall that a group is hyper-FC-central if and only if none of its quotients has the infinite conjugacy class (ICC) property. The result of [FrischHartmanTamuzVahidi Ferdowsi, 2019] is further developed in [ErschlerKaimanovich, 2023, Theorem A], where it is proved that any countable ICC group G admits symmetric non-degenerate measures, such that the associated non-trivial Poisson boundary can be described in terms of the convergence to the boundary of a locally finite forest, whose vertex set is G. All of the correspondences mentioned above fail if we restrict ourselves to probability measures with a finite first moment. Indeed, the group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^3$ is amenable, but every non-degenerate measure with a finite first moment (and more generally, with finite entropy) has a non-trivial Poisson boundary. Additionally, finitely generated groups of intermediate growth provide examples of ICC groups in which every probability measure with a finite first moment has a trivial Poisson boundary.

The known descriptions of the Poisson boundary for non-amenable groups often rely on the group possessing some hyperbolic-like nature. This is the case for free groups [Derriennic, 1975; DynkinMaljutov, 1961], hyperbolic groups [Ancona, 1987], [Kaimanovich, 2000, Theorems 7.4 and 7.7], [ChawlaForghaniFrischTiozzo, 2022, Theorem 1.1], and more generally acylindrically hyperbolic groups [MaherTiozzo, 2018, Theorem 1.5], [ChawlaForghaniFrischTiozzo, 2022, Theorem 1.2]. These papers cover previously known descriptions for some specific families of groups; see [FarbMasur, 1998; GauteroMathéus, 2012; Horbez, 2016; KaimanovichMasur, 1996; Woess, 1989]. Another family of groups in this category are discrete subgroups of semi-simple Lie groups [Furstenberg, 1971; Ledrappier, 1985], [Kaimanovich, 2000, Theorems 10.3 and 10.7].

The most relevant descriptions of (non-trivial) Poisson boundaries for our results are those of wreath products, which provided the first examples of amenable groups that admit symmetric random walks with a non-trivial Poisson boundary [KaimanovichVershik, 1983, Proposition 6.1]. We now make a summary of the known complete descriptions of Poisson boundaries of wreath products.

– The first complete description of a non-trivial Poisson boundary of a random walk on a wreath product was by James-Peres [JamesPeres, 1996, Corollary 1.1], who considered some particular degenerate measures on $\mathbb{Z} \wr \mathbb{Z}^d$, $d \geq 1$. In general, one can associate with every random walk on a finitely generated group B a degenerate random walk on $\mathbb{Z} \wr B$ that uses the lamp group $\mathbb{Z}$ to keep track of the number of visits to each point in the base group B. Then, the Poisson boundary of this random walk on $\mathbb{Z} \wr B$ is equal to the *exchangeability boundary* of the induced random walk on B. James and Peres proved that the Poisson boundary of these specially chosen measures on $\mathbb{Z} \wr \mathbb{Z}^d$ is completely described by the number of visits to points in the base group. We refer to [JamesPeres, 1996], [Kaimanovich, 1991, Section 6] and [Erschler, 2011, Subsection 4.1] for the definition of the exchangeability boundary and more details on this correspondence. James and Peres' description of the Poisson boundary of $\mathbb{Z} \wr \mathbb{Z}^d$ for such degenerate random walks is based on the fact that transient random walks on $\mathbb{Z}^d$

with a finitely supported step distribution have infinitely many cutpoints with probability 1 [JamesPeres, 1996, Theorem 1.2].

- In the case of non-degenerate measures, Kaimanovich proved the following: for any probability measure on $A \wr \mathbb{Z}^d$, $d \geq 1$, with a finite first moment and such that the induced random walk on $\mathbb{Z}^d$ has non-zero drift, the Poisson boundary is equal to the space of limit lamp configurations, endowed with the corresponding hitting measure [Kaimanovich, 2001, Theorem 3.6.6]. This result is a consequence of the strip criterion, whose hypotheses are verified by using the fact that the induced random walk on the base group $\mathbb{Z}^d$ has a linear rate of escape. This technique does not work if the induced random walk on $\mathbb{Z}^d$ is centered (which can only occur for $d \geq 3$ in the case of non-trivial Poisson boundaries).
- There are other base groups, distinct from $\mathbb{Z}^d$, for which the strip criterion was applied to provide a description of the Poisson boundary for non-degenerate measures with a finite first moment in the corresponding wreath product: this has been done for finite lamp groups A and for B a (non-abelian) free group [KarlssonWoess, 2007, Theorem 3.2], and for B non-elementary hyperbolic or with infinitely many ends [Sava, 2010, Theorems 4.2 and 4.3].
- Returning to the case where the base group is $\mathbb{Z}^d$, Erschler proved that for A finitely generated, $d \geq 5$, and for a probability measure μ on $A \wr \mathbb{Z}^d$ with a finite third moment and that induces a transient random walk on $\mathbb{Z}^d$, the associated Poisson boundary is equal to the space of limit lamp configurations, endowed with the hitting measure [Erschler, 2011, Theorem 1]. Her proof consists of using the support of the limit lamp configuration together with cut-balls of the trajectory in the base group to guess which points of it are visited at certain time instants. With this, the identification of the Poisson boundary is obtained by using a version of Kaimanovich's ray criterion.
- The latter result was extended by Lyons-Peres to $d \geq 3$, and for measures μ with a finite second moment and that induce a transient random walk on $\mathbb{Z}^d$ [LyonsPeres, 2021a, Theorem 5.1]. In their proof, the second moment assumption controls how often lamps are modified very far away from the current position, and an argument with cut-spheres together with Kaimanovich's conditional entropy criterion concludes the proof. In addition, they show that the space of limit lamp configurations describes the Poisson boundary of random walks on groups $A \wr B$ with A finite and B finitely generated, for step distributions with finite entropy, that induce a transient random walk on the base group B, and with bounded lamp range (i.e. at each step of the random walk, the lamp configuration φ_n is modified in a uniformly bounded neighborhood of the current base position X_n) [LyonsPeres, 2021a, Theorem 1.1]. Lyons-Peres also prove an enhanced version of Kaimanovich's conditional entropy criterion [LyonsPeres, 2021a, Corollary 2.3], and use it to give a short proof of the description of the Poisson boundary for simple random walks on $A \wr \mathbb{Z}^d$, $d \geq 3$, by using cut-spheres to estimate the values of the lamp configuration at a given instant [LyonsPeres, 2021a, Theorem 3.1]. In particular, the results of Lyons-Peres cover the description of the Poisson boundary for simple random walks on groups $A \wr B$ where A is finite and B is finitely generated, and recover the results that had been obtained previously in the literature for finitely supported random walks.

Our method for proving Theorem 5.1.3 is different from the results cited previously. Since there are no moment assumptions in our hypotheses, the estimates we obtain for the lamp configuration at some given instant rely uniquely on the fact that stabilization does occur. Indeed, for the (possibly heavy-tailed) measures we consider, there may be frequent modifications to the lamp configuration at distances arbitrarily far away from the base position. For these measures, the method of cut-balls used by Erschler for measures with a finite third moment does not apply, and neither does the technique of cut-spheres used by Lyons-Peres for measures with a finite second moment. We also mention that our proof uses the original version of the conditional entropy criterion of Kaimanovich, and does not require the enhanced version proved by Lyons-Peres. The technique we use to prove our theorems has some commonalities with the *pin-down approximation* used by [ChawlaForghaniFrischTiozzo, 2022] for (acylindrically) hyperbolic groups. In both arguments, the entropy of the random walk position at some time instant is estimated by conditioning on partitions of the space of trajectories that add a small amount of information. The core part of the proof of [ChawlaForghaniFrischTiozzo, 2022, Theorems 1.1 and 1.2] is estimating the word length of the random walk element at a given instant, and it relies on the fact that a sample path of the random walk on a hyperbolic group can be divided into consecutive segments, each separated from the next one by a *pivot* of the trajectory. The position of the random walk is then deduced from the additional partitions associated with the time instants in which pivots occur, together with the distance between consecutive pivots and the increments from long time intervals in which there were no pivots. In our case, the argument is different. The most important part of our proof corresponds to Steps (7)–(10) in the sketch below, where the stabilization of lamp configurations allows us to reveal the lamp increments that were done in group elements whose associated lamp configuration has not yet stabilized to its limit value, at a given time instant.

We now sketch the proof of our main theorems.

5.1.3 Sketch of the proof of Theorems 5.1.3 and 5.1.6

Thanks to Kaimanovich's conditional entropy criterion (Theorem 5.2.4), in order to prove Theorem 5.1.3 (resp. Theorem 5.1.6) it suffices to show that for every $\varepsilon > 0$, the mean conditional entropy of the random walk conditioned on hitting boundary points $(\varphi_\infty, \xi) \in A^B \times \partial B$ (resp. $\varphi_\infty \in A^B$) is at most $\varepsilon n + K$, where K is a constant independent of n. The idea is that, conditioned on the boundary point (φ_∞, ξ) (resp. φ_∞), we can recover the position of the random walk at time n by adding information in the form of countable partitions of the space of trajectories, that in average add a small amount of entropy to our process.

For every $\varepsilon > 0$, we follow the next steps.

1. The first step of the proof is to observe that the boundary point allows us to determine the position in B of the μ-random walk every t_0 steps, for t_0 large enough, at the cost of adding a small amount of entropy. We save this information in the partition $\mathcal{P}_n^{t_0}$ and call it the t_0-*coarse trajectory* (Definition 5.3.3). In the case of Theorem 5.1.3, the above is done by conditioning on the boundary point $\xi \in \partial B$ (Lemma 5.3.4), whereas for Theorem 5.1.6 we do it by conditioning on the limit lamp configuration φ_∞, and use the additional

hypothesis of a finite first moment (Proposition 5.5.2). This is the only place where there is a difference between the proofs of Theorems 5.1.3 and 5.1.6, and the rest of the sketch below is common to both of them.

2. The partition $\mathcal{P}_n^{t_0}$ contains the position in B in the last instant that is a multiple of t_0. Then, getting to the time instant n adds only a constant amount of entropy. Because of this, throughout the rest of the sketch we will assume that n is a multiple of t_0, so that we do not have to mention this argument multiple times.

   In what follows, we explain our approach to estimating the entropy of the lamp configuration at time n.

3. We introduce the notion of (R, L)-*good elements*, which are the group elements that only do movements in the base group or lamp modifications inside some finite neighborhood $R \subseteq B$ of the current position in B, and which modify lamp configurations by a state inside some finite set $L \subseteq A$ (Definition 5.3.6). An element is said to be (R, L)-*bad* if it is not (R, L)-good. Note that if A and B are finitely generated, then so is $A \wr B$, and our definition of an element being (R, L)-good is equivalent to having a uniform upper bound (depending on R and L) for its word length.

4. We divide the time interval $[0, n]$ into subintervals I_j, $j = 1, \ldots, n/t_0$, of length t_0 (Definition 5.3.5). We say that an interval I_j is (R, L)-*bad* if some increment g_i, $i \in I_j$, is (R, L)-bad (Definition 5.3.6). We save the information of the increments done during an (R, L)-bad interval in the partition $\beta_n(t_0, R, L)$, and call it the collection of (t_0, R, L)-*bad increments* (Definition 5.3.8).

5. We then prove that when R and L are large enough, the partition $\beta_n(t_0, R, L)$ has small entropy (Lemma 5.3.9). From this, we can estimate the values of the lamp configuration at instant n for the elements of B as follows.

6. Define the *coarse neighborhood* $\mathcal{N}_n(t_0, R)$ of the trajectory as a neighborhood of $\mathcal{P}_n^{t_0}$ (in the base group B) such that any modification to the lamp configuration outside of $\mathcal{N}_n(t_0, R)$ must have occurred during an (R, L)-bad interval (Definition 5.3.11). Then the values of the lamp configuration at elements outside $\mathcal{N}_n(t_0, R)$ can be deduced from the t_0-coarse trajectory $\mathcal{P}_n^{t_0}$ together with the information of the (t_0, R, L)-bad increments $\beta_n(t_0, R, L)$ (Lemma 5.3.13).

7. The most important part of the proof is to prove that the entropy of the lamp configuration inside the coarse neighborhood $\mathcal{N}_n(t_0, R)$ is low when conditioned on the limit lamp configuration φ_∞, the t_0-coarse trajectory $\mathcal{P}_n^{t_0}$ and the (t_0, R, L)-bad increments $\beta_n(t_0, R, L)$ (Proposition 5.4.1). Here is where we use the hypothesis that the lamp configuration stabilizes almost surely. This is the objective of Section 5.4. In Subsection 5.4.1 we present a proof of Proposition 5.4.1 in the case where the lamp group A is finite. The case of infinite lamp groups has some additional difficulties, and the idea of our proof is the following.

8. In expectation, the lamp configuration of most of the elements inside the t_0-coarse neighborhood $\mathcal{P}_n^{t_0}$ will be stabilized to the value assigned by the limit lamp configuration φ_∞ (Lemma 5.4.3). Hence, revealing the ones that are not yet stabilized has low entropy

(Lemma 5.4.5). Furthermore, if t_0 is large enough, then revealing the instants where these increments were applied also has low entropy (Lemma 5.4.8).

9. If these increments were done during an (R, L)-bad interval, we already know their value since this information is contained in the partition $\beta_n(t_0, R, L)$. On the other hand, if they were done during an (R, L)-good interval, we can bound the possible number of values they can take in terms of the size of R and L. With this, we see that the (R, L)-good lamp increments done at every position whose lamp configuration has not yet stabilized contain a small amount of entropy (Lemma 5.4.10).

10. The lamp configuration inside the coarse neighborhood $\mathcal{N}_n(t_0, R)$ is completely determined by the above increments together with the information of the (t_0, R, L)-bad intervals $\beta_n(t_0, R, L)$ and the t_0-coarse trajectory $\mathcal{P}_n^{t_0}$. Together with Step (6), we conclude that the mean conditional entropy with respect to the limit lamp configuration φ_∞, conditioned on the t_0-coarse trajectory $\mathcal{P}_n^{t_0}$ and the (t_0, R, L)-bad increments $\beta_n(t_0, R, L)$ has low entropy (Proposition 5.5.1).

5.1.4 Organization

In Section 5.2 we recall some generalities on Poisson boundaries and entropy. Then in Section 5.3 we give the setup for Theorem 5.1.3 and prove the entropy estimates of Steps (1)–(6) for partitions of the space of trajectories up to instant n mentioned in the sketch of the proof above. The most important part of our argument is the entropy estimate described in Steps (7)–(10) above, and it is presented in Section 5.4. The key result of this section is Proposition 5.4.1. To prove it, we introduce additional partitions related to the group elements for which the lamp configuration has not stabilized to its limit value by time n, and which do not play a role outside this section. Next, in Section 5.5 we present the proofs of Theorems 5.1.3 and 5.1.6, as well as Corollary 5.1.5. Finally, in Section 5.6 we prove Corollaries 5.1.8 and 5.1.9 for groups of the form $F_d/[N, N]$ and free solvable groups.

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5.2 Preliminaries

5.2.1 Wreath products and lamplighter groups

For A, B groups, we define their *wreath product* $A \wr B$ as the semidirect product $\bigoplus_B A \rtimes B$, where $\bigoplus_B A$ is the group of finitely supported functions $f : B \to A$ endowed with the operation $\oplus$ of componentwise multiplication. We denote by $\mathrm{supp}(f)$ the finite subset of B to which f assigns non-trivial values. Here, the group B acts on the direct sum $\bigoplus_B A$ from the left by translations. That is, for $f : B \to A$, and any $b \in B$ we have

$$(b \cdot f)(x) = f(b^{-1}x), \ x \in B.$$

Elements of $A \wr B$ can be expressed as a tuple (f, b), where $f \in \bigoplus_B A$ and $b \in B$. The product between two elements elements (f, b), $(f', b') \in A \wr B$ is given by

$$(f, b) \cdot (f', b') = (f \oplus (b \cdot f'), bb').$$

There is a natural embedding of B into $A \wr B$ via the mapping

$$\begin{aligned} B &\to A \wr B \\ b &\mapsto (\mathbb{1}, b), \end{aligned}$$

where $\mathbb{1}(x) = e_A$ for any $x \in B$. Similarly, we can embed A into $A \wr B$ via the mapping

$$\begin{aligned} B &\to A \wr B \\ a &\mapsto (\delta^a_{e_B}, e_B), \end{aligned}$$

where $\delta^a_{e_B}(e_B) = a$ and $\delta^a_{e_B}(x) = e_A$ for any $x \neq e_B$. In particular, if A and B are finitely generated, one can consider finite generating sets S_A and S_B of A and B, respectively, and their copies inside $A \wr B$ through the above embeddings generate the entire group $A \wr B$.

Wreath products of the form $\mathbb{Z}/2\mathbb{Z} \wr B$ are also called "lamplighter groups", due to the following interpretation: one pictures the Cayley graph $\mathrm{Cay}(B, S_B)$ as a street with lamps at every element, each of which can be in a different independent state given by an element of $\mathbb{Z}/2\mathbb{Z}$. The identity of $\mathbb{Z}/2\mathbb{Z}$ corresponds to the lamp being turned off, and the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ corresponds to the lamp being turned on. An element $g = (f, b) \in \mathbb{Z}/2\mathbb{Z} \wr B$ consists of a *lamp configuration* $f \in \bigoplus_B \mathbb{Z}/2\mathbb{Z}$, which encodes the state of the lamp at each element of B, together with a position $b \in B$, which corresponds to a person standing next to a particular lamp. Multiplying on the right by elements of $\mathbb{Z}/2\mathbb{Z}$ corresponds to switching the state of the lamp at position b, whereas multiplying on the right by an element of B changes the position of the person, without modifying the lamp configuration.

5.2.2 Random walks and the Poisson boundary

Let G be a countable group and let μ be a probability measure on G. The *μ-random walk* on G is the Markov chain $\{w_n\}_{n\geq 0}$ defined by $w_0 = e_G$, and for $n \geq 1$,

$$w_n = g_1 g_2 \cdots g_n,$$

where $\{g_i\}_{i\geq 1}$ is a sequence of independent, identically distributed, random variables on G with law μ. The law $\mathbb{P}$ of the process $\{w_n\}_{n\geq 1}$ is defined as the push-forward of the Bernoulli measure $\mu^{\mathbb{N}}$ through the map

$$\begin{aligned} G^{\mathbb{N}} &\to G^{\mathbb{N}} \\ (g_1, g_2, g_3, \ldots) &\mapsto (w_1, w_2, w_3, \ldots) \coloneqq (g_1, g_1 g_2, g_1 g_2 g_3, \ldots). \end{aligned}$$

The space $(G^{\mathbb{N}}, \mathbb{P})$ is called the *path space* or the *space of trajectories* of the μ-random walk.

The Poisson boundary of (G, μ) is a probability measure space (B, ν) that is endowed with

1. a measurable action of G, and
2. a G-equivariant map $\mathbf{bnd} : (G^{\mathbb{N}}, \mathbb{P}) \to (B, \nu)$ such that $\mathbf{bnd} \circ T = \mathbf{bnd}$ and $\nu = \mathbf{bnd}_* \mathbb{P}$.

These conditions imply that ν is a *μ-stationary* measure, meaning that it satisfies the equation $\nu = \mu * \nu \coloneqq \sum_{g\in G} \mu(g) g_* \nu$. In general, any probability space (X, λ) that satisfies conditions (1) and (2) is called a *μ-boundary* of G. The Poisson boundary is unique up to a G-equivariant isomorphism and it is maximal, in the sense that the boundary map $(G^{\mathbb{N}}, \mathbb{P}) \to (X, \lambda)$ of any μ-boundary (X, λ) factors through (B, ν).

In this paper we will work with Definition 5.1.1 for the Poisson boundary. Alternatively, it can also be defined as the space of ergodic components of the shift map $T : G^{\mathbb{N}} \to G^{\mathbb{N}}$ on the space of trajectories, where $T(w_1, w_2, w_3, \ldots) \coloneqq (w_2, w_3, \ldots)$ for $(w_1, w_2, \ldots) \in G^{\mathbb{N}}$. For further equivalent definitions the Poisson boundary, we refer to [KaimanovichVershik, 1983], [Kaimanovich, 2000, Section 1] and the references therein. We also refer to the surveys [Erschler, 2010; Furman, 2002] for an overview of the study of Poisson boundaries and to the survey [Zheng, 2023] for more recent applications of random walks and Poisson boundaries to group theory.

5.2.3 Entropy

The entropy of a probability measure μ on the group G is defined as $H(\mu) \coloneqq -\sum_{g\in G} \mu(g) \log(\mu(g))$. More generally, let $\rho = \{\rho_k\}_{k\geq 1}$ be a countable partition of the space of sample paths $G^{\mathbb{N}}$. The entropy of ρ with respect to the measure $\mathbb{P}$ is defined as

$$H(\rho) \coloneqq -\sum_{k\geq 1} \mathbb{P}(\rho_k) \log \mathbb{P}(\rho_k).$$

The following sequence of partitions will appear in the formulation of Theorem 5.2.3.

Definition 5.2.1. For every $n \geq 1$, define the partition α_n of the space of sample paths $G^{\mathbb{N}}$, where two trajectories $\mathbf{w}, \mathbf{w}' \in G^{\mathbb{N}}$ belong to the same element of α_n if and only if $w_n = w'_n$.

That is, α_n is the partition given by the n-th instant of the μ-random walk. Note that we have

$$H(\alpha_n) = -\sum_{g\in G} \mathbb{P}(w_n = g) \log \mathbb{P}(w_n = g) = H(\mu^{*n}),$$

so that the entropy of the partition α_n coincides with the entropy of the convolution μ^{*n}, which is the law of the n-th step of the μ-random walk on G. From this, the asymptotic entropy h_μ of the random walk (G, μ) (Definition 5.1.2) can be expressed in terms of the partitions α_n, $n \geq 1$, as $h_\mu = \lim_{n\to\infty} \frac{H(\alpha_n)}{n}$.

The *entropy criterion* of Derriennic [Derriennic, 1980] and Kaimanovich-Vershik [KaimanovichVershik, 1983] states that if μ is a probability measure on G with $H(\mu) < \infty$, then the Poisson boundary of (G, μ) is trivial if and only if $h_\mu = 0$. In what follows we recall the basic definitions of the conditional probability measures and conditional entropy associated with a μ-boundary, and then state the *conditional entropy criterion* of Kaimanovich below, that characterizes whether a μ-boundary is the Poisson boundary.

5.2.4 The conditional entropy criterion

Let $\mathbf{X} = (X, \lambda)$ be a μ-boundary of G. Using Rokhlin's theory of measurable partitions of Lebesgue spaces, the probability measure $\mathbb{P}$ can be disintegrated with respect to the map $(G^{\mathbb{N}}, \mathbb{P}) \to (X, \lambda)$. That is, for λ-a.e. $\xi \in X$, there exists the conditional probability measure $\mathbb{P}^\xi$ on $G^{\mathbb{N}}$, and it holds that $\mathbb{P} = \int_X \mathbb{P}^\xi \, d\lambda(\xi)$; see [Rohlin, 1967, Section I.7] and [Kaimanovich, 2000, Section 3].

Definition 5.2.2. Consider $\rho = \{\rho_k\}_{k\geq 1}$ a countable partition of the space of sample paths $G^{\mathbb{N}}$ and let $\mathbf{X} = (X, \lambda)$ be a μ-boundary of G. For λ-a.e. $\xi \in X$, we define the *conditional entropy of ρ given ξ* as

$$H_\xi(\rho) := -\sum_{k\geq 1} \mathbb{P}^\xi(\rho_k) \log \mathbb{P}^\xi(\rho_k).$$

We now state Kaimanovich's conditional entropy criterion, which is one of the main tools to determine whether a given μ-boundary is equal to the Poisson boundary of the μ-random walk on G.

Theorem 5.2.3 ([Kaimanovich, 1985, Theorem 2], [Kaimanovich, 2000, Theorem 4.6])**.** *Let G be a countable group, and let μ be a probability measure on G with $H(\mu) < \infty$. Consider a μ-boundary $\mathbf{X} = (X, \lambda)$ of G. Then $\mathbf{X}$ is the Poisson boundary of (G, μ) if and only if for λ-almost every $\xi \in X$ we have*

$$\lim_{n\to\infty} \frac{H_\xi(\alpha_n)}{n} = 0.$$

Here α_n is the partition associated with the value of the μ-random walk at instant n (Definition 5.2.1).

Following [Rohlin, 1967, Section 5.1], let us define the *mean conditional entropy over the μ-boundary* $\mathbf{X}$ as

$$H_{\mathbf{X}}(\rho) := \int_X H_\xi(\rho) \, d\lambda(\xi).$$

The mean conditional entropy with respect to the Poisson boundary of (G, μ) has been considered in the proof of the entropy criterion for the triviality of the Poisson boundary in [KaimanovichVershik, 1983] (see Equation (12) at the bottom of page 464). For arbitrary μ-boundaries, the mean conditional entropy is considered in [Kaimanovich, 1985] in the proof of the conditional entropy criterion (see the paragraph preceding Theorem 2 in page 194).

Another way of stating Kaimanovich's conditional entropy criterion (Theorem 5.2.3) is the following, which uses the mean conditional entropy.

Theorem 5.2.4. *Let G be a countable group, and let μ be a probability measure on G with $H(\mu) < \infty$. Consider a μ-boundary $\mathbf{X} = (X, \lambda)$ of G. Then $\mathbf{X}$ is the Poisson boundary of (G, μ) if and only if*

$$\lim_{n \to \infty} \frac{H_{\mathbf{X}}(\alpha_n)}{n} = 0.$$

The mean conditional entropy can be obtained by taking the logarithm of the transition probabilities of the conditional random walk, and then integrating over the path space (see Equation (20) on page 462 of [KaimanovichVershik, 1983]). Thus, the formulation of the conditional entropy criterion in Theorem 5.2.4 follows from Theorem 5.2.3. We also mention that Theorem 5.2.4 is a particular case of [KaimanovichSobieczky, 2012, Theorem 2.17], where the result is proved in a more general setting for random walks along classes of graphed equivalence relations. For random walks on groups, the equivalence relation that one considers is the one induced by identifying sample paths of $G^{\mathbb{N}}$ which are mapped to the same boundary point $\xi \in X$. We thank Anna Erschler and Vadim Kaimanovich for pointing out to us the paper [Kaimanovich, 1985], and for many explanations about the results that appear on it.

In the proofs of our main results (Theorems 5.1.3 and 5.1.6) we will use the formulation of Kaimanovich's conditional entropy criterion in terms of the mean conditional entropy (Theorem 5.2.4).

5.2.5 Properties about entropy

It will be important for our proofs that the mean conditional entropy of a partition over a μ-boundary is at most the entropy of the partition. We state this in the following lemma.

Lemma 5.2.5. *Consider a countable partition ρ of the space of sample paths $G^{\mathbb{N}}$. Let $\mathbf{X}$ be a μ-boundary of G and $\mathbf{Y}$ a quotient of $\mathbf{X}$ with respect to a G-equivariant measurable partition. Then it holds that $H_{\mathbf{X}}(\rho) \le H_{\mathbf{Y}}(\rho)$.*

In particular for every countable partition ρ and each μ-boundary $\mathbf{X}$, we have that $H_{\mathbf{X}}(\rho) \le H(\rho)$.

This follows from the fact that the function $x \mapsto -x \log(x)$ is concave on $[0, 1]$, together with Jensen's inequality and the fact that conditional entropies are non-negative (see [Rohlin, 1967, Section 5.10]).

We finish this section by introducing the notion of conditional entropy with respect to a partition. Consider two countable partitions ρ, γ of the path space $G^{\mathbb{N}}$. Let (X, λ) be a μ-

boundary of G. For every $\xi \in X$, denote

$$\mathbb{P}^{\xi}(P \mid Q) \coloneqq \frac{\mathbb{P}^{\xi}(P \cap Q)}{P^{\xi}(Q)}, \text{ for } P \in \rho \text{ and } Q \in \gamma \text{ with } P^{\xi}(Q) \neq 0.$$

We define the associated *conditional entropy of* ρ *with respect to* γ *conditioned on* ξ as

$$H_{\xi}(\rho \mid \gamma) \coloneqq - \sum_{P \in \rho, Q \in \gamma} \mathbb{P}^{\xi}(P \cap Q) \log \mathbb{P}^{\xi}(P \mid Q).$$

Define also

$$H_{\mathbf{X}}(\rho \mid \gamma) \coloneqq \int_X H_{\xi}(\rho \mid \gamma)\ d\lambda(\xi).$$

We have the equality $H_{\mathbf{X}}(\rho \mid \gamma) = H_{\mathbf{X}}(\rho \vee \gamma) - H_{\mathbf{X}}(\gamma)$, where $\rho \vee \gamma \coloneqq \{P \cap Q \mid P \in \rho, Q \in \gamma\}$ is the *join* of the partitions ρ and γ.

Remark 5.2.6. Rokhlin's theory of measurable partitions [Rohlin, 1967] implies that for every μ-boundary $\mathbf{X}$ of G, there is an associated measurable partition η of the space of sample paths $G^{\mathbb{N}}$. With this, one can express the mean conditional entropy of a countable partition ρ over the μ-boundary $\mathbf{X}$ as $H_{\mathbf{X}}(\rho) = H(\rho \mid \eta)$. Similarly, for two countable partitions ρ, γ of the space of sample paths, we have $H_{\mathbf{X}}(\rho \mid \gamma) = H(\rho \mid \gamma \vee \eta)$. This is the notation used in [Kaimanovich, 1985; KaimanovichVershik, 1983] when the mean conditional entropy is discussed.

Throughout this paper we will use the notation $H_{\mathbf{X}}(\rho \mid \gamma)$ for the mean conditional entropy, in order to emphasize the dependence on the μ-boundary that is being considered.

In the proofs of our results we will use the following basic facts about entropy.

Lemma 5.2.7. *Consider countable partitions* ρ, γ *and* δ *of* $G^{\mathbb{N}}$*, and a* μ*-boundary* $\mathbf{X}$*. The following properties hold.*

1. $H_{\mathbf{X}}(\rho \vee \gamma \mid \delta) = H_{\mathbf{X}}(\rho \mid \gamma \vee \delta) + H_{\mathbf{X}}(\gamma \mid \delta)$.
2. $H_{\mathbf{X}}(\rho \mid \gamma) \leq H_{\mathbf{X}}(\rho \vee \delta \mid \gamma)$.
3. $H_{\mathbf{X}}(\rho \mid \gamma \vee \delta) \leq H_{\mathbf{X}}(\rho \mid \gamma)$.

We refer to [MartinEngland, 1981, Corollaries 2.5 and 2.6] and [Rohlin, 1967, Section 5] for the proofs of these properties, and more generally to [MartinEngland, 1981, Chapter 2] for more details about entropy.

Remark 5.2.8. In order to simplify notation, throughout this paper we will use the same symbol to denote both a random variable and the partition of the space of sample paths that it defines. More precisely, let $X : (G^{\mathbb{N}}, \mathbb{P}) \to D$ be a random variable from the space of sample paths with values on a countable set D. Then the countable partition ρ_X of $(G^{\mathbb{N}}, \mathbb{P})$ defined by X is given by saying that two sample paths $\mathbf{w}, \mathbf{w}' \in G^{\mathbb{N}}$ belong to the same element of ρ_X if and only if $X(\mathbf{w}) = X(\mathbf{w}')$. Then, we will denote $H(X) \coloneqq H(\rho_X)$.

5.3 Setup and the coarse trajectory

5.3.1 Definition of the boundary

Let A and B be countable groups, and consider their wreath product $A \wr B = \bigoplus_B A \rtimes B$ endowed with a probability measure μ. Let us denote the μ-random walk on $A \wr B$ as

$$w_n \coloneqq (\varphi_n, X_n) = (f_1, Y_1) \cdots (f_n, Y_n),\ n \geq 1,$$

where $g_i \coloneqq (f_i, Y_i)$, $i \geq 1$, is a sequence of independent increments with law μ.

The μ-random walk on $A \wr B$ naturally projects to a random walk on B. Namely, denote by $\pi_B : A \wr B \to B$ the canonical projection, and consider $\mu_B \coloneqq \pi_* \mu$. Then $\{X_n\}_{n \geq 0}$ is a μ_B-random walk on B. Let us denote by $(\partial B, \nu_B)$ the Poisson boundary of the random walk (B, μ_B). Note that $(\partial B, \nu_B)$ is also an $(A \wr B, \mu)$-boundary, and denote by $\mathbf{bnd}_B : (A \wr B)^\infty \to \partial B$ the associated boundary map.

Definition 5.3.1. We say that the lamp configurations of the μ-random walk $\{(\varphi_n, X_n)\}_{n \geq 0}$ on $A \wr B$ stabilize almost surely if for every $b \in B$, there exists $N \geq 1$ such that $\varphi_n(b) = \varphi_N(b)$ for every $n \geq N$.

The hypothesis that lamp configurations stabilize almost surely implies that there is a map $\mathcal{L} : (A \wr B)^\infty \to A^B$ that assigns to $\mathbb{P}$-almost every trajectory $w_n = (\varphi_n, X_n)$, $n \geq 1$, the limit value φ_∞ of the lamp configurations φ_n, $n \geq 1$. Let us denote $\nu_{\mathcal{L}} \coloneqq \mathcal{L}_* \mathbb{P}$ the associated hitting measure. Since the map $\mathcal{L}$ is shift-invariant, the measure $\nu_{\mathcal{L}}$ is μ-stationary, and the space $(A^B, \nu_{\mathcal{L}})$ is an $(A \wr B, \mu)$-boundary.

The map

$$\begin{aligned} (A \wr B)^\infty &\to A^B \times \partial B \\ w &\mapsto (\mathcal{L}(w), \mathrm{bnd}_B(w)), \end{aligned}$$

pushes forward $\mathbb{P}$ to a μ-stationary measure ν on $A^B \times \partial B$. With this, $(A^B \times \partial B, \nu)$ is a μ-boundary of $A \wr B$. Theorem 5.1.3 claims that in this setting, this space is the Poisson boundary of the μ-random walk on $A \wr B$.

5.3.2 Transience

The following (not difficult) lemma states that the stabilization of lamp configurations implies that the projected random walk on B is transient, as long as the step distribution has a non-trivial projection to the lamps group $\bigoplus_B A$. This observation goes back to the original paper of Kaimanovich-Vershik [KaimanovichVershik, 1983], and we provide its proof for the convenience of the reader.

Lemma 5.3.2. *Let μ be a probability measure on $A \wr B$ such that lamp configurations stabilize almost surely. Assume that* $\mathrm{supp}(\mu)$ *is not completely contained in B. Then μ induces a transient random walk on B.*

Proof. Indeed, suppose that the projection of the random walk to B is recurrent. Consider an element $g = (f, x) \in A \wr B$ with $\mu(g) > 0$, such that $\mathrm{supp}(f) \neq \varnothing$. Choose $b \in B$ such that $f(b) \neq e_A$. Then the Borel-Cantelli Lemma [Feller, 1968, Lemma 2 of Section VIII.3] implies that the lamp configuration at b will change infinitely many times along almost every trajectory of the μ-random walk. □

Note that if the induced random walk on B is recurrent, then Lemma 5.3.2 implies that the support of μ must be completely contained in B. In particular, the space $(A^B, \nu_{\mathcal{L}})$ is trivial, since no modification to the lamp coordinate is possible. In that case, $(A^B \times \partial B, \nu)$ is isomorphic to $(\partial B, \nu_B)$, and the conclusion of Theorem 5.1.3 follows.

5.3.3 Main Assumptions

From now on and throughout the rest of the paper, we will assume that $H(\mu) < \infty$, that lamp configurations stabilize almost surely, and that the induced random walk on B is transient.

5.3.4 The coarse trajectory in the base group

The following corresponds to Step (1) of the sketch of the proof (Subsection 5.1.3).

Definition 5.3.3 (Coarse trajectory)**.** Let $t_0 \geq 1$. We define the t_0-*coarse trajectory* on the base group B of the μ-random walk by

$$\mathcal{P}_n^{t_0} := \left(X_{t_0}, X_{2t_0}, \dots, X_{\lfloor n/t_0 \rfloor t_0}\right) \in B^{\lfloor n/t_0 \rfloor}.$$

That is, $\mathcal{P}_n^{t_0}$ is the ordered tuple of length $\lfloor n/t_0 \rfloor$ formed by the random variables X_{it_0}, $i = 1, \dots, \lfloor n/t_0 \rfloor$, that correspond to the projections of the μ-random walk to B every t_0 steps.

The following lemma tells us that if t_0 is large enough, then the t_0-coarse trajectory $\mathcal{P}_n^{t_0}$ has low entropy when conditioned on the Poisson boundary $(\partial B, \nu_B)$ of the base group B. The statement below is written in terms of the mean conditional entropy (Definition 5.2.2) over the μ_B-boundary ∂B.

Lemma 5.3.4. *Let $\varepsilon > 0$. Then there exists $T \geq 1$ such that for every $t_0 \geq T$,*

$$H_{\partial B}(\mathcal{P}_n^{t_0}) < \varepsilon n, \text{ for any } n \geq 1.$$

Proof. Let $\varepsilon > 0$. Then the conditional entropy criterion (Theorem 5.2.4) implies that there exists $T \geq 1$ such that for every $t_0 \geq T$, we have $H_{\partial B}(X_{t_0}) < \varepsilon t_0$.

Let $n \geq 1$, $t_0 \geq T$, and denote $s = \lfloor n/t_0 \rfloor$. Using Item 1 of Lemma 5.2.7, we have that

$$H_{\partial B}(X_{t_0}, X_{2t_0}, X_{3t_0}, \dots, X_{st_0}) = H_{\partial B}(X_{t_0}) + H_{\partial B}(X_{2t_0}, X_{3t_0}, \dots, X_{st_0} \mid X_{t_0}).$$

Additionally, using Items 1 and 3 of Lemma 5.2.7 we have

$$H_{\partial B}(X_{2t_0}, X_{3t_0}, \dots, X_{st_0} \mid X_{t_0}) = H_{\partial B}(X_{2t_0} \mid X_{t_0}) + H_{\partial B}(X_{3t_0}, \dots, X_{st_0} \mid X_{t_0}, X_{2t_0})$$

$$\leq H_{\partial B}(X_{2t_0} \mid X_{t_0}) + H_{\partial B}(X_{3t_0}, \dots, X_{st_0} \mid X_{2t_0}),$$

so that

$$H_{\partial B}(X_{t_0}, X_{2t_0}, X_{3t_0}, \dots, X_{st_0}) \leq H_{\partial B}(X_{t_0}) + H_{\partial B}(X_{2t_0} \mid X_{t_0}) + H_{\partial B}(X_{3t_0}, \dots, X_{st_0} \mid X_{2t_0}).$$

By repeating this argument $s-1$ times, we obtain

$$H_{\partial B}(X_{t_0}, X_{2t_0}, \dots, X_{st_0}) \leq \sum_{j=0}^{s-1} H_{\partial B}\left(X_{(j+1)t_0} \mid X_{jt_0}\right).$$

Finally, using the above together with the fact that $H_{\partial B}\left(X_{(j+1)t_0} \mid X_{jt_0}\right) = H_{\partial B}(X_{t_0})$ for every $j = 0, \dots, s-1$, we get

$$H_{\partial B}(X_{t_0}, X_{2t_0}, \dots, X_{st_0}) \leq sH_{\partial B}(X_{t_0}) < s\varepsilon t_0 \leq \varepsilon n.$$

□

5.3.5 Good and bad intervals

We now divide the time interval between 0 and n into subintervals of length t_0. Afterward, we define what it means for these intervals to be *good* or *bad*, and introduce the partitions of bad increments. This corresponds to steps (3)–(5) of the sketch of the proof (Subsection 5.1.3).

Definition 5.3.5. Consider $t_0 \geq 1$. For $j = 1, 2, \dots, \lfloor n/t_0 \rfloor$, let us define the *$j$-th interval* $I_j := \{(j-1)t_0 + 1, \dots, jt_0\}$, and the *final interval* $I_{\text{final}} := \{\lfloor n/t_0 \rfloor t_0 + 1, \dots, n\}$.

Note that if n is a multiple of t_0, then $I_{\text{final}} = \varnothing$.

Recall that we denote by $\{g_i\}_{i\geq 1}$ the sequence of (independent, identically distributed) increments of the μ-random walk on $A \wr B$.

Definition 5.3.6. Let us consider finite subsets $R \subseteq B$ and $L \subseteq A$, with $e_B \in R$ and $e_A \in L$.

- Say that a group element $g = (f, x)$ is *(R, L)-good* if $x \in R$, $\operatorname{supp}(f) \subseteq R$ and $f(b) \in L$ for every $b \in B$. Otherwise, call g an *(R, L)-bad* element.
- For each $i = 1, 2, \dots, \lfloor n/t_0 \rfloor$, say that the interval I_j is *(R, L)-good* if all the increments g_i, $i \in I_j$, are (R, L)-good. Otherwise, let us say that the interval I_j is *(R, L)-bad.*

Let us fix throughout the paper the auxiliary symbol $\star$. This symbol will play the role of unknown information, and it will serve as a tool to obtain low entropy events from high entropy ones. This will be carried out using the following general lemma about entropy, which is also stated in [ChawlaForghaniFrischTiozzo, 2022, Lemma 2.4]. It is a standard result in entropy theory and we provide its proof for the convenience of the reader.

Lemma 5.3.7 (Obscuring lemma)**.** *Let D be a countable set, and fix an element $\star \notin D$. Let $X : (\Omega, \mathbb{P}) \to D$ be a random variable with $H(X) < \infty$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$*

such that for any measurable set $E \subseteq \Omega$ with $\mathbb{P}(E) < \delta$, the random variable

$$\widetilde{X}(\omega) := \begin{cases} X(\omega) & \text{if } \omega \in E, \text{ and} \\ \star & \text{if } \omega \notin E, \end{cases}$$

satisfies $H(\widetilde{X}) < \varepsilon$.

Intuitively, this lemma says that a random variable with finite entropy can be modified into a random variable with arbitrarily small entropy, at the cost of obscuring its values in a set of large measure. This lemma will be used below in the proof of Lemma 5.3.9 to estimate entropy of the increments that occurred during (R, L)-bad intervals, for R and L large enough.

Proof. Let us denote $\kappa : [0, 1] \to [0, 1]$ the function $\kappa(x) := -x \log(x)$, where we use the convention $0 \log(0) = 0$. The function κ is continuous, concave, strictly increasing in the interval $[0, 1/e]$ and strictly decreasing in the interval $[1/e, 1]$.

Let $\varepsilon > 0$. Since X has finite entropy, we can find a finite subset $Q \subseteq D$ such that $\sum_{d \in D \setminus Q} \kappa\left(\mathbb{P}(X = d)\right) < \varepsilon/2$.

Let us choose $0 < \delta < 1/e$ small enough such that $\kappa(1 - \delta) + |Q|\kappa(\delta) < \varepsilon/2$. Such δ exists thanks to the continuity of κ and the fact that $\kappa(0) = \kappa(1) = 0$. Consider an arbitrary measurable subset $E \subseteq \Omega$ with $\mathbb{P}(E) < \delta$ and consider $\widetilde{X}$ defined as in the formulation of the lemma. Then we have

$$\begin{aligned} H(\widetilde{X}) &= \kappa(\mathbb{P}(\Omega \setminus E)) + \sum_{d \in Q} \kappa(\mathbb{P}(\{X = d\} \cap E)) + \sum_{d \in D \setminus Q} \kappa(\mathbb{P}(\{X = d\} \cap E)) \\ &\leq \kappa(1 - \delta) + \sum_{d \in Q} \kappa(\mathbb{P}(E)) + \sum_{d \in D \setminus Q} \kappa(\mathbb{P}(X = d)) \\ &= \kappa(1 - \delta) + |Q|\kappa(\delta) + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

□

Definition 5.3.8. Let $t_0 \geq 1$, and consider finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$. For every $j = 1, 2, \ldots, \lfloor n/t_0 \rfloor$, define the random variable

$$Z_j := \begin{cases} (g_i)_{i \in I_j}, & \text{if } I_j \text{ is an } (R, L)\text{-bad interval, and} \\ \star, & \text{otherwise,} \end{cases}$$

which has values in the space $(A \wr B)^{I_j} \cup \{\star\}$. Let us also define $Z_{\text{final}} := (g_i)_{i \in I_{\text{final}}}$, which has values in the space $(A \wr B)^{I_{\text{final}}}$. We define the (t_0, R, L)-*bad increments* by

$$\beta_n(t_0, R, L) := (Z_1, Z_2, \ldots, Z_{\lfloor n/t_0 \rfloor}, Z_{\text{final}}).$$

In other words, $\beta_n(t_0, R, L)$ is an ordered tuple formed by $\lfloor n/t_0 \rfloor + 1$ random variables that are defined as follows: each of the first $\lfloor n/t_0 \rfloor$ variables correspond to either the ordered list of increments g_i, $i \in I_j$ that occurred during an (R, L)-bad interval I_j, or the symbol $\star$ in the case where I_j was an (R, L)-good interval. The last position of the tuple $\beta_n(t_0, R, L)$ contains the ordered list of increments that occurred during the final time interval $I_{\text{final}} = \{\lfloor n/t_0 \rfloor + 1, \lfloor n/t_0 \rfloor + 2, \ldots, n\}$.

The following lemma tells us that if R and L are large enough, then the tuple of (R, L)-bad increments has low entropy.

Lemma 5.3.9. *Let $t_0 \geq 1$. For any $\varepsilon > 0$, there exist finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$, and $K_1 \geq 0$ such that for every $n \geq 1$,*

$$H(\beta_n(t_0, R, L)) < \varepsilon n + K_1.$$

Proof. Consider $t_0 \geq 1$ and $\varepsilon > 0$. First let us note that I_{final} consists of at most t_0 instants, and hence there is a constant $K_1 \geq 0$, that only depends on μ and t_0, such that $H(Z_{\text{final}}) \leq K_1$.

Let us introduce the random variables W_j, $1 \leq j \leq \lfloor n/t_0 \rfloor$, defined by $W_j = (g_i)_{i \in I_j}$. That is, W_j corresponds to the increments of the μ-random walk that occurred during the interval I_j. Note that the variables W_j are identically distributed and have finite entropy. By using Lemma 5.3.7, we can find $\delta > 0$ such that if $E \subseteq (A \wr B)^{t_0}$ satisfies $\mu^{t_0}(E) < \delta$, then the variable

$$\widetilde{W}_j = \begin{cases} (g_i)_{i \in I_j}, & \text{if } E \text{ occurs, and} \\ \star, & \text{otherwise,} \end{cases}$$

satisfies $H(\widetilde{W}_j) < \varepsilon$.

Choose finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$, such that

$$\mu^{t_0}(I_1 \text{ is an } (R, L)\text{-bad interval}) < \delta.$$

For every $1 \leq j \leq \lfloor n/t_0 \rfloor$, consider the random variable $\widetilde{W}_j$, associated with the event $E_j \coloneqq \{I_j \text{ is an } (R, L)\text{-bad interval}\}$. Then $H(\widetilde{W}_j) < \varepsilon$, and we have $Z_j = \widetilde{W}_j$. With this, we conclude that

$$H(\beta_n(t_0, R, L)) \leq \sum_{j=1}^{\lfloor n/t_0 \rfloor} H(Z_j) + Z_{\text{final}} < \varepsilon n + K_1, \quad \text{for every } n \geq 1.$$

□

In particular whenever the lamp group A is finite, one can choose $L = A$ in Lemma 5.3.9.

The following lemma says that the trajectory at time n can be deduced from the trajectory at the last time instant that is a multiple of t_0, together with the partition of (t_0, R, L)-bad intervals. The statement is expressed in terms of the mean conditional entropy (Definition 5.2.2) with respect to the boundary of infinite lamp configurations A^B.

Lemma 5.3.10. *Let $n \geq 1$, $t_0 \geq 1$ and consider finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$. For every $n \geq 1$,*

$$H_{A^B}\left(\varphi_n, X_n \mid \varphi_s \vee \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) = 0, \text{ where we denote } s = \lfloor n/t_0 \rfloor t_0.$$

Proof. The position (φ_s, X_s) of the μ-random walk at time s is completely determined by φ_s and $\mathcal{P}_n^{t_0}$, which contains the value of X_s. Furthermore, $\beta_n(t_0, R, L)$ contains the increments Z_{final} of the last interval. With this information, we can completely determine (φ_n, X_n). □

5.3.6 The coarse neighborhood

Now we introduce the coarse neighborhood of the trajectory and do the entropy estimates of Step (6) of the sketch of the proof (Subsection 5.1.3)

Definition 5.3.11. Let $t_0 \geq 1$ and $R \subseteq B$ be a finite subset with $e_B \in R$. Define the (t_0, R)-*coarse neighborhood* of the trajectory at instant n by

$$\mathcal{N}_n(t_0, R) \coloneqq \bigcup_{j=0}^{\lfloor n/t_0 \rfloor - 1} \{X_{jt_0} r_1 r_2 \cdots r_{t_0} \mid r_k \in R, \text{ for } k = 1, 2, \dots, t_0\}.$$

In order to estimate the entropy of the n-th instant of the random walk, it will be useful to divide the values of the lamp configuration into the ones inside the coarse neighborhood and the ones outside of it.

Definition 5.3.12. Consider $t_0 \geq 1$ and a finite subset $R \subseteq B$ with $e_B \in R$. We define the *lamp configuration inside the* (t_0, R)*-coarse neighborhood* $\mathcal{N}_n(t_0, R)$ as

$$\Phi_n^{\text{in}}(t_0, R) \coloneqq \varphi_n|_{\mathcal{N}_n(t_0, R)},$$

and the *lamp configuration outside the* (t_0, R)*-coarse neighborhood* $\mathcal{N}_n(t_0, R)$ as

$$\Phi_n^{\text{out}}(t_0, R) \coloneqq \varphi_n|_{B \setminus \mathcal{N}_n(t_0, R)}.$$

The following lemma says that the lamp configuration outside the coarse neighborhood is determined by the coarse trajectory together with the increments that occurred during the bad intervals.

Lemma 5.3.13. *Let $t_0 \geq 1$, and let $R \subseteq B$, $L \subseteq A$ be finite subsets with $e_B \in R$, $e_A \in L$. Consider any $n \geq 1$ and denote $s = \lfloor n/t_0 \rfloor t_0$. Then*

$$H\left(\Phi_s^{\text{out}}(t_0, R) \middle| \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) = 0.$$

Proof. Consider t_0, R and L as above. Note that any modification of the lamp configuration φ_s at an element $b \notin \mathcal{N}_n(t_0, R)$ must have occurred during an (R, L)-bad interval. Since we condition on the information of the (t_0, R, L)-bad increments and $\mathcal{P}_n^{t_0}$, we know the exact value of every (R, L)-bad increment and the position on B of the μ-random walk at the moment that it was applied. This completely determines the value of φ_s in every element of $B \setminus \mathcal{N}_n(t_0, R)$. □

The entropy estimates for the lamp configuration inside the coarse neighborhood are carried out in the following section.

5.4 Entropy inside the coarse neighborhood

The objective of this section is to prove the following proposition, which states that the entropy of the lamp configuration inside the coarse neighborhood is low when conditioned on the limit lamp configuration and the increments that occurred during the bad intervals. The statement below is expressed in terms of the mean conditional entropy (Definition 5.2.2) with respect to the boundary of infinite lamp configurations A^B. This corresponds to Step (7) of the sketch of the proof (Subsection 5.1.3).

Proposition 5.4.1. *For any $\varepsilon > 0$ there exists $T \geq 1$ such that the following holds. Consider any finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$, and $t_0 \geq T$. Then there exist $N \geq 1$ and $K \geq 0$ such that*

$$H_{A^B}\left(\Phi_s^{\text{in}}(t_0, R) \middle| \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) < \varepsilon n + K, \tag{5.1}$$

for every $n \geq N$, where we denote $s = \lfloor n/t_0 \rfloor t_0$.

This result is proved in Subsection 5.4.5, for which we first prove some additional lemmas in Subsections 5.4.2, 5.4.3 and 5.4.4. In the case where the lamp group A is finite, Proposition 5.4.1 can be proved directly, without passing through intermediate lemmas. This proof is presented in Subsection 5.4.1 to expose the main ideas that go into the proof of Proposition 5.4.1, and does not play a role in the proof of the general case where the lamp group A may be infinite in Subsection 5.4.5.

5.4.1 Proof of Proposition 5.4.1 when the lamp group A is finite

Proof. Let $\varepsilon > 0$, and let us consider $T = 1$. Consider any finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$, and let $t_0 \geq T$.

Recall that we assume that the lamp configuration φ_n, $n \geq 1$, of the μ-random walk on $A \wr B$ stabilizes almost surely to an infinite lamp configuration φ_∞. Let us define, for each $n \geq 1$ and $i \in \left\{0, 1, \dots, \lfloor \frac{n}{t_0} \rfloor - 1\right\}$, the event

$$E_{i,n} := \left\{\text{there exists } b \in X_{it_0} R^{t_0} \text{ such that } \varphi_n(b) \neq \varphi_\infty(b)\right\}.$$

In other words, $E_{i,n}$ is the event where there is some lamp configuration on the R^{t_0}-neighborhood of X_{it_0} (in B) that is distinct at time n from its limit value.

Let us fix $\delta > 0$ that satisfies

$$\delta < \frac{\varepsilon}{2|R^{t_0}| \log(|A|)}, \text{ and } -\delta \log(\delta) - (1-\delta)\log(1-\delta) < \varepsilon/2. \tag{5.2}$$

The hypothesis of stabilization along sample paths implies that there exists $N \geq 1$ such that for all $n \geq N$ we have $\mathbb{P}(E_{0,s}) < \delta$, where $s = \lfloor n/t_0 \rfloor t_0$. Note that thanks to the Markov property and the group invariance of the process, for all $i \in \left\{0, 1, \dots, \lfloor \frac{n}{t_0} \rfloor - 1\right\}$ we have

$$\mathbb{P}(E_{i,s}) = \mathbb{P}(E_{0,s-it_0}).$$

Together with the above, for each $n \geq N$ and $i = 0, 1, \dots, \left\lfloor \frac{n-N+1}{t_0} \right\rfloor$ we have that $\mathbb{P}(E_{i,s}) < \delta$.

Let us now prove that for all $n \geq N$, we have

$$H_{A^B}\left(\Phi_s^{\mathrm{in}}(t_0, R) \middle| \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) < \varepsilon n + K,$$

where $s = \lfloor n/t_0 \rfloor t_0$. In order to do so, it suffices to prove that for all $n \geq N$ and $i = 0, 1, \dots, \left\lfloor \frac{n-N+1}{t_0} \right\rfloor$ we have

$$H_{A^B}\left(\varphi_s|_{N_i} \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) < \varepsilon, \tag{5.3}$$

where we denote $N_i \coloneqq X_{it_0} R^{t_0}$ the R^{t_0}-neighborhood of X_{it_0}. Indeed, $\varphi_s|_{\mathcal{N}_n(t_0,R)}$ is completely determined by the values $\varphi_s|_{N_i}$, with $i = 0, 1, \dots, \left\lfloor \frac{n}{t_0} \right\rfloor$. Thanks to Equation (5.3), the terms corresponding to $i = 0, 1, \dots, \left\lfloor \frac{n-N+1}{t_0} \right\rfloor$ each contribute an amount of ε to the entropy of $\varphi_s|_{\mathcal{N}_n(t_0,R)}$, whereas the remaining values $i = \left\lfloor \frac{n-N+1}{t_0} \right\rfloor + 1, \dots, \left\lfloor \frac{n}{t_0} \right\rfloor$ contribute at most $K \coloneqq H\left(\mu^{*(N+1+t_0)}\right)$.

In order to finish the proof, let us show that Equation (5.3) holds. For each $i = 0, 1, \dots, \left\lfloor \frac{n-N+1}{t_0} \right\rfloor$, denote by $\mathcal{E}_i = \{E_{i,s}, E_{i,s}^c\}$ the partition of the space of sample paths induced by the event $E_{i,s}$ defined above. Note that we have that $H(\mathcal{E}_i) \leq -\delta \log(\delta) - (1-\delta)\log(1-\delta) < \varepsilon/2$, since $\mathbb{P}(E_{i,s}) < \delta$ for δ as in Equation (5.2).

In order to estimate the value of $H_{\varphi_\infty}\left(\varphi_s|_{N_i} \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L) \vee \mathcal{E}_i\right)$, we remark that in the event $E_{i,s}^c$, the lamp configuration at time s coincides with the limit lamp configuration φ_∞. This implies that for each $\varphi_\infty \in A^B$ we have

$$\begin{aligned} H_{\varphi_\infty}\left(\varphi_s|_{N_i} \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L) \vee \mathcal{E}_i\right) &\leq \log\left(|A|^{|N_i|}\right) \mathbb{P}(E_{i,s}) \\ &\leq \log\left(|A|^{|N_i|}\right) \delta. \end{aligned}$$

Now we integrate with respect to $\nu_{\mathcal{L}}$ over all $\varphi_\infty \in A^B$, and we get

$$H_{A^B}\left(\varphi_s|_{N_i} \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L) \vee \mathcal{E}_i\right) \leq \log\left(|A|^{|N_i|}\right) \delta.$$

By noticing that $|N_i| = |R^{t_0}|$ and thanks to our choice of δ as in Equation (5.2), we see that $\log\left(|A|^{|N_i|}\right) \delta < \varepsilon/2$. From this, using the general properties of entropy from Lemma 5.2.7 together with what we just showed, we conclude that

$$\begin{aligned} H_{A^B}\left(\varphi_s|_{N_i} \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) &\leq H_{A^B}\left(\varphi_s|_{N_i} \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L) \vee \mathcal{E}_i\right) + H(\mathcal{E}_i) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This proves Equation (5.3). ☐

In the following subsections we prove Proposition 5.4.1 in its full generality, where the lamp group A may be infinite.

5.4.2 Unstable points in the base group

Recall that we denote by φ_∞ the limit lamp configuration associated with a trajectory $\{(\varphi_n, X_n)\}_{n\geq 0}$ of the μ-random walk on $A \wr B$. We now define our notion of *unstable points* at a given time instant, which are the positions on B for which the lamp configuration has not yet stabilized to its limit value.

Definition 5.4.2. Let $t_0 \geq 1$ and consider finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$. For $n \geq 1$, let us define the set of *unstable points at time* $s := \lfloor n/t_0 \rfloor t_0$ as

$$\mathcal{U}_s := \{b \in \mathcal{N}_n(t_0, R) \mid \varphi_s(b) \neq \varphi_\infty(b)\}.$$

The next lemma says that, in expectation, there are not too many unstable points.

Lemma 5.4.3. *Let $t_0 \geq 1$ and consider finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$. For any $\varepsilon > 0$, there exists a constant $K_2 > 0$ such that for every $n \geq 1$,*

$$\mathbb{E}\left(|\mathcal{U}_s| \mid \mathcal{P}_n^{t_0}\right) < \varepsilon n + K_2, \text{ where we denote } s = \lfloor n/t_0 \rfloor t_0.$$

Proof. Consider t_0, R and L as above. For $0 \leq j \leq \lfloor n/t_0 \rfloor - 1$, let us denote $F_j := X_{jt_0} R^{t_0}$ and recall that $\mathcal{N}_n(t_0, R) = \bigcup_{j=0}^{\lfloor n/t_0 \rfloor - 1} F_j$ (Definition 5.3.11).

For $n_0 \geq 1$, let us say that an instant jt_0 is *poorly n_0-stabilized* if the lamp configuration at F_j is modified at some instant beyond $jt_0 + n_0$. That is, if for some $k > jt_0 + n_0$ and some $b \in F_j$, we have $\varphi_{k-1}(b) \neq \varphi_k(b)$.

We observe the following.

Claim 1: *For every $n_0 \geq 1$ and $j \geq 0$, we have*

$$\mathbb{P}(j \textit{ is poorly } n_0\textit{-stabilized}) = \mathbb{P}(0 \textit{ is poorly } n_0\textit{-stabilized}).$$

Indeed, we see that

$$\begin{aligned}\mathbb{P}(j \text{ is poorly } n_0\text{-stabilized}) &= \mathbb{P}(\varphi_{k-1}(b) \neq \varphi_k(b) \text{ for some } k > jt_0 + n_0 \text{ and } b \in X_{jt_0} R^{t_0}) \\ &= \mathbb{P}(\varphi_{k-1}(b) \neq \varphi_k(b) \text{ for some } k > n_0 \text{ and } b \in R^{t_0}) \\ &= \mathbb{P}(0 \text{ is poorly } n_0\text{-stabilized}).\end{aligned}$$

Claim 2: $\lim_{n_0\to\infty} \mathbb{P}(0$ *is poorly n_0-stabilized*$) = 0$.

Indeed, this follows from the hypothesis that lamp configurations stabilize almost surely, which implies that

$$\mathbb{P}(\text{the lamp configuration in } R^{t_0} \text{ is modified after time } n_0) \xrightarrow[n_0\to\infty]{} 0.$$

Let $\varepsilon > 0$. Using Claims 1 and 2 we can find $n_0 \geq 1$ such that for every $k \geq n_0$ and $j \geq 0$,

$$\mathbb{P}(j \text{ is poorly } k\text{-stabilized}) \leq \mathbb{P}(j \text{ is poorly } n_0\text{-stabilized}) < \varepsilon.$$

Thanks to the linearity of expectation, we have

$$\begin{aligned}\mathbb{E}\left(|\mathcal{U}_s| \mid \mathcal{P}_n^{t_0}\right) &= \mathbb{E}\left(\sum_{j=0}^{\lfloor n/t_0\rfloor -1} \mathbb{1}_{\left\{\varphi_s|_{F_j} \neq \varphi_\infty|_{F_j}\right\}} \mid \mathcal{P}_n^{t_0}\right)\\ &= \sum_{j=0}^{\lfloor n/t_0\rfloor -1} \mathbb{P}\left(\varphi_s|_{F_j} \neq \varphi_\infty|_{F_j} \mid \mathcal{P}_n^{t_0}\right)\\ &= \sum_{j=0}^{\lfloor n/t_0\rfloor -1} \mathbb{P}(j \text{ is poorly } s\text{-stabilized}).\end{aligned}$$

Denote $J = \lfloor \lfloor n/t_0 \rfloor - n_0/t_0 - 1\rfloor$, which is the largest integer that satisfies the inequality $Jt_0 + t_0 \leq \lfloor n/t_0\rfloor t_0 - n_0$. In other words, Jt_0 is the last instant whose associated interval I_{Jt_0} still has n_0 remaining units of time before reaching time s. Then if $j \in \{0, 1, \dots, J\}$, it holds that $\mathbb{P}(j \text{ is poorly } s\text{-stabilized}) \leq \mathbb{P}(0 \text{ is poorly } n_0\text{-stabilized})$. From this, we see that

$$\begin{aligned}\mathbb{E}\left(|\mathcal{U}_s| \mid \mathcal{P}_n^{t_0}\right) &\leq \sum_{j=0}^{J} \mathbb{P}(j \text{ is poorly } s\text{-stabilized}) + \sum_{j=J+1}^{\lfloor n/t_0\rfloor -1} \mathbb{P}(j \text{ is poorly } s\text{-stabilized})\\ &\leq (J+1)\mathbb{P}(0 \text{ is poorly } n_0\text{-stabilized}) + \lfloor n/t_0\rfloor - 1 - J\\ &< (J+1)\varepsilon + \lfloor n/t_0\rfloor - 1 - J.\end{aligned}$$

Our choice of J guarantees that $J+1 \leq (n-n_0)/t_0 \leq n$ and $\lfloor n/t_0\rfloor - 1 - J \leq n_0/t_0 + 1$, so that

$$\mathbb{E}\left(|\mathcal{U}_s| \mid \mathcal{P}_n^{t_0}\right) \leq \varepsilon n + K_2,$$

where $K_2 = n_0/t_0 + 1$ is a constant that only depends on μ, t_0, R and ε. ☐

Now we will prove that the small expectation of the number of unstable points implies that revealing their values is a low entropy event. For this, we will need a general lemma about entropy, whose proof we present below for the convenience of the reader.

Lemma 5.4.4. *Let D be a non-empty finite set. Let Z be a random subset of D. That is, a random variable that takes values in the power set of D. Suppose that $\mathbb{E}(|Z|) \leq \alpha|D|$ for some $0 \leq \alpha \leq 1/2$. Then*

$$H(Z) \leq |D|(-\alpha\log\alpha - (1-\alpha)\log(1-\alpha)).$$

Proof. Let us define the binary entropy function $h(p) \coloneqq -p\log p - (1-p)\log(1-p)$, for $0 \leq p \leq 1$. The value $h(p)$ coincides with the entropy of a Bernoulli random variable X with values in $\{0,1\}$, and with $\mathbb{P}(X=1) = p$. It is well-known that for a given $0 \leq p \leq 1$ and every random variable Q that takes values in a two-element set $\{a, b\}$ with $\mathbb{P}(Q = a) = p$, it holds that $H(Q) \leq h(p)$. Furthermore, the function $h(p)$ is concave, and it is strictly increasing in the interval $0 \leq p \leq 1/2$.

Let us now prove the lemma. For every $d \in D$, let us define $Z_d \coloneqq \mathbb{1}_{\{d\in Z\}}$. Note that Z_d is a Bernoulli random variable with parameter $p_d \coloneqq \mathbb{P}(d \in Z)$. Then Z is completely determined by $\{Z_d\}_{d\in D}$, and we have $\mathbb{E}(|Z|) = \sum_{d\in D} p_d \leq \alpha|D|$, for a fixed $0 \leq \alpha \leq 1/2$.

First, we note that

$$H(Z) = H(\{Z_d\}_{d\in D}) \leq \sum_{d\in D} H(Z_d) \leq \sum_{d\in D} h(p_d).$$

Now, using Jensen's inequality, since h is concave and $\alpha \leq 1/2$, we get

$$\sum_{d\in D} h(p_d) = |D| \sum_{d\in D} \frac{1}{|D|} h(p_d) \leq |D| h\left(\sum_{d\in D} \frac{1}{|D|} p_d\right) \leq |D| h\left(\frac{\mathbb{E}(|Z|)}{|D|}\right) \leq |D| h(\alpha).$$

□

Lemma 5.4.5 (The set of unstable points has small entropy)**.** *Let $t_0 \geq 1$ and consider finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$. For every $\varepsilon > 0$, there exists $N_1 \geq 1$ such that $H(\mathcal{U}_s \mid \mathcal{P}_n^{t_0}) < \varepsilon n$ for any $n \geq N_1$, where we denote $s = \lfloor n/t_0 \rfloor t_0$.*

Proof. Let us consider t_0, R and L as above. Since $\mathcal{U}_s$ is a subset of $\mathcal{N}_n(t_0, R)$, we know that the size of $\mathcal{U}_s$ is at most $n|R|^{t_0}$. Consider an arbitrary $\varepsilon > 0$, and let us consider $\widetilde{\varepsilon} > 0$ to be fixed later. Using Lemma 5.4.3 we can find a constant $K_2 \geq 0$ such that

$$\mathbb{E}\left(|\mathcal{U}_s| \mid \mathcal{P}_n^{t_0}\right) < \widetilde{\varepsilon} n + K_2 = \frac{t_0\widetilde{\varepsilon} + K_2 \frac{t_0}{n}}{|R^{t_0}|} \frac{n}{t_0} |R^{t_0}|.$$

Now we are going to use Lemma 5.4.4. Let us define $M_n := \frac{n}{t_0}|R^{t_0}|$ and let D_n be an enumeration of the coarse trajectory $\mathcal{U}_s$ with fixed symbols from 1 up to M_n. This is a codification that gives rise to an injective map from subsets of the coarse neighborhood to subsets of indices in D_n. By virtue of the data processing inequality, we have that $H(\mathcal{U}_s \mid \mathcal{P}_n^{t_0}) \leq H(f(\mathcal{U}_s))$. We proved above that for $\alpha = \frac{t_0\widetilde{\varepsilon} + K_2 \frac{t_0}{n}}{|R^{t_0}|}$, it holds that $\mathbb{E}(|f(\mathcal{U}_s)|) < \alpha |D_n|$. If $\widetilde{\varepsilon}$ is small enough and n is large enough, then $\alpha < 1/2$ and we can apply Lemma 5.4.4 to show that

$$H\left(\mathcal{U}_s \mid \mathcal{P}_n^{t_0}\right) \leq H(f(\mathcal{U}_s)) \leq \frac{n}{t_0}|R|^{t_0}(-\alpha \log \alpha - (1-\alpha)\log(1-\alpha)).$$

The statement of the lemma follows by choosing $\widetilde{\varepsilon}$ small enough and N_1 large enough such that $-\alpha \log \alpha - (1-\alpha)\log(1-\alpha) < \varepsilon t_0/|R^{t_0}|$. □

5.4.3 Visits to unstable points

We now give estimates for the entropy associated with knowing the instants up to time n where there could have been a lamp modification at an unstable point by a good element. Lemmas 5.4.3, 5.4.5 and 5.4.8 below correspond to Step (8) of the sketch of the proof (Subsection 5.1.3).

Definition 5.4.6. Let $t_0 \geq 1$ and consider finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$. Let $n \geq 1$ and denote $s = \lfloor n/t_0 \rfloor t_0$. Let us define the times of *visit to unstable points* as

$$\mathcal{V}_s = \{j \in \{1,2,\dots,\lfloor n/t_0 \rfloor\} \mid I_j \text{ is an } (R,L)\text{-good interval and } X_{(j-1)t_0}R^{t_0} \cap \mathcal{U}_s \neq \varnothing\}.$$

Lemma 5.4.7. *Let $t_0 \geq 1$ and consider finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$. For any $\varepsilon > 0$, there exists $K_3 \geq 1$ such that for every $n \geq 1$,*

$$\mathbb{E}\left(|\mathcal{V}_s| \mid \mathcal{P}_n^{t_0}\right) < \varepsilon n + K_3, \text{ where we denote } s = \lfloor n/t_0 \rfloor t_0.$$

Proof. Let t_0, R and L be as above. For an arbitrary $\ell \geq 1$, define

$$J_\ell := \{j \in \mathbb{N} \mid \text{ there exist } n_1 > n_2 > \ldots > n_\ell > (j-1)t_0 \text{ such that } X_{n_k} = X_{(j-1)t_0}, \text{ for } k = 1,2,\dots,\ell\}.$$

In other words, the set J_ℓ is formed by the values $j \geq 1$ such that the element $X_{(j-1)t_0}$ is visited at least ℓ times more in the future.

Claim 1: For every $j \geq 1$, we have $\mathbb{P}(1 \in J_\ell) = \mathbb{P}(j \in J_\ell)$.

Indeed, by using the Markov property, we see that for every $j \geq 1$,

$$\begin{aligned}\mathbb{P}(j \in J_\ell) &= \mathbb{P}(X_{(j-1)t_0} \text{ is visited more than } \ell \text{ times in the future}) \\ &= \mathbb{P}(e_B \text{ is visited more than } \ell \text{ times in the future}) \\ &= \mathbb{P}(1 \in J_\ell).\end{aligned}$$

Claim 2: We have $\lim_{\ell\to\infty} \mathbb{P}(1 \in J_\ell) = 0$.
Indeed, recall that we assume that the induced random walk on B is transient (see Subsection 5.3.3 and the paragraph above it). The claim is then just a consequence of the transience of this random walk.

Claim 3: $\mathbb{E}(|\mathcal{V}_s \backslash J_\ell| \mid \mathcal{P}_n^{t_0}) \leq \ell |R^{t_0}| \mathbb{E}(|\mathcal{U}_s| \mid \mathcal{P}_n^{t_0})$.
Indeed, note that

$$\begin{aligned}\mathbb{E}\left(|\mathcal{V}_s \backslash J_\ell| \mid \mathcal{P}_n^{t_0}\right) &\leq \mathbb{E}\left(\sum_{j=1}^{\lfloor n/t_0 \rfloor} \mathbb{1}_{\{X_{(j-1)t_0}R^{t_0}\cap \mathcal{U}_s \neq \varnothing\}} \mathbb{1}_{\left\{\substack{X_{(j-1)t_0} \text{ is visited } \leq \ell \\ \text{times after instant } jt_0}\right\}} \middle| \mathcal{P}_n^{t_0}\right) \\ &= \mathbb{E}\left(\sum_{u \in \mathcal{U}_s} \sum_{r \in R^{t_0}} \sum_{j=1}^{\lfloor n/t_0 \rfloor} \mathbb{1}_{\{X_{(j-1)t_0} = ur^{-1}\}} \mathbb{1}_{\left\{\substack{X_{(j-1)t_0} \text{ is visited } \leq \ell \\ \text{times after instant } jt_0}\right\}} \middle| \mathcal{P}_n^{t_0}\right) \\ &\leq \mathbb{E}\left(|\mathcal{U}_s||R^{t_0}|\ell \middle| \mathcal{P}_n^{t_0}\right) \\ &= \ell |R^{t_0}| \mathbb{E}(|\mathcal{U}_s| \mid \mathcal{P}_n^{t_0}).\end{aligned}$$

Consider $\varepsilon > 0$ arbitrary. Thanks to Claim 2, we can find $\ell \geq 1$ large enough such that

$$p_\ell := \mathbb{P}(0 \in J_\ell) < \frac{\varepsilon}{2t_0}.$$

By virtue of Lemma 5.4.3, we can find $K_2 \geq 0$ such that $\mathbb{E}(|\mathcal{U}_s| \mid \mathcal{P}_n^{t_0}) < \frac{\varepsilon}{2\ell|R^{t_0}|}n + K_2$..

Now, using Claims 1 and 3, we see that

$$\begin{aligned}\mathbb{E}\left(|\mathcal{V}_s| \mid \mathcal{P}_n^{t_0}\right) &= \mathbb{E}\left(|J_\ell| \mid \mathcal{P}_n^{t_0}\right) + \mathbb{E}\left(|\mathcal{V}_s \backslash J_\ell| \mid \mathcal{P}_n^{t_0}\right) \\ &\leq \sum_{j=1}^{t_0} \mathbb{P}(j \in J_\ell) + \ell|R^{t_0}|\mathbb{E}(|\mathcal{U}_s| \mid \mathcal{P}_n^{t_0}) \\ &< t_0 p_\ell + \ell|R^{t_0}| \left(\frac{\varepsilon}{2\ell|R^{t_0}|}n + K_2\right) \\ &< \varepsilon n + \ell K_2.\end{aligned}$$

The statement of the lemma follows from setting $K_3 := K_2\ell|R^{t_0}|$, which is a constant that depends only on μ, t_0, R and ε. □

The next lemma states that listing the visits to unstable points is a low entropy event.

Lemma 5.4.8. *Let $\varepsilon > 0$. Then there exists $T \geq 1$ such that the following holds. For any finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$, and $t_0 \geq T$, it holds that $H(\mathcal{V}_s) < \varepsilon n$ for every $n \geq 1$, where we denote $s = \lfloor n/t_0 \rfloor t_0$.*

Proof. Let $\varepsilon > 0$. For any R and L as above and an arbitrary value of t_0, the visit times to unstable points $\mathcal{V}_s$ form a subset of $\{1, 2, \ldots, \lfloor n/t_0 \rfloor\}$. Then $H(\mathcal{V}_s) \leq \frac{n}{t_0}\log(2)$, so that it suffices to choose $T > \frac{\log(2)}{\varepsilon}$ at the beginning and $t_0 \geq T$. □

5.4.4 Lamp increments at unstable elements

So far, we have proved that, conditioned on the t_0-coarse trajectory $\mathcal{P}_n^{t_0}$, it is possible to reveal the positions in the base group for which the lamp configuration has not yet stabilized, together with the time instants in which the lamp state at these positions could have been modified, while adding a small amount of entropy to our process. We will now see that the same holds for the values of the lamp increments that modified these positions in these time instants.

Definition 5.4.9. Let $t_0 \geq 1$, and consider finite subsets $R \subseteq B, L \subseteq A$ with $e_B \in R$ and $e_A \in L$. Let $j \in \{1, 2, \ldots, \lfloor n/t_0 \rfloor\}$ and $i \in I_j$. Let $n \geq 1$ and denote $s = \lfloor n/t_0 \rfloor t_0$. For $b \in \mathcal{U}_s$, let us define the *unstable increment in b at instant i* by

$$\Delta_s(b, i) := \begin{cases} \varphi_i(b)^{-1}\varphi_{i+1}(b), & \text{if } I_j \text{ is an } (R, L)\text{-good interval, and} \\ \star, \text{ otherwise.} \end{cases}$$

The next lemma says that, conditioned on knowing the set of unstable elements $\mathcal{U}_s$ together with the instants of visits $\mathcal{V}_s$ to their neighborhood, revealing the lamp increments made to these elements is a low entropy event. This is Step (9) of the sketch of the proof (Subsection 5.1.3).

Lemma 5.4.10. *Let $t_0 \geq 1$, and consider finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$. For every $\varepsilon > 0$ there exists $K_4 \geq 0$ such that for every $n \geq 1$,*

$$H(\Delta_s \mid \mathcal{P}_n^{t_0} \vee \mathcal{U}_s \vee \mathcal{V}_s) < \varepsilon n + K_4, \text{ where we denote } s = \lfloor n/t_0 \rfloor t_0.$$

Proof. Let us consider t_0, R and L as above. Consider $\varepsilon > 0$. Note that for every $1 \leq j \leq \lfloor n/t_0 \rfloor$, $i \in I_j$ and $b \in \mathcal{U}_s$, the variable $\Delta_s(b,i)$ can take a total of at most $|L|^{|R^{t_0}|}+1$ different values. Using Lemma 5.4.7 we can find $K_3 \geq 0$ such that

$$\mathbb{E}(|\mathcal{V}_s| \mid \mathcal{P}_n^{t_0}) < \frac{\varepsilon n}{\log\left(|L|^{|R^{t_0}|}+1\right)} + K_3.$$

Now we have

$$\begin{aligned} H(\Delta_s \mid \mathcal{P}_n^{t_0} \vee \mathcal{U}_s \vee \mathcal{V}_s) &\leq \mathbb{E}(|\mathcal{V}_s| \mid \mathcal{P}_n^{t_0}) \log\left(|L|^{|R^{t_0}|}+1\right) \\ &\leq \varepsilon n + K_3 \log\left(|L|^{|R^{t_0}|}+1\right). \end{aligned}$$

The statement of the lemma follows by choosing $K_4 := K_3 \log\left(|L|^{|R^{t_0}|}+1\right)$. □

5.4.5 Proof of Proposition 5.4.1

Now we prove the main result of this section.

Proof of Proposition 5.4.1. Let $\varepsilon > 0$. Using Lemma 5.4.8 we can find $T \geq 1$ such that for any R and L as above, if $t_0 > T$, then $H(\mathcal{V}_s) < \frac{\varepsilon}{3} n$ for every $n \geq 1$. Furthermore, let us consider a constant $K_4 \geq 0$ such that

$$H(\Delta_s \mid \mathcal{P}_n^{t_0} \vee \mathcal{U}_s \vee \mathcal{V}_s) < \frac{\varepsilon}{3} n + K_4$$

for every $n \geq 1$, which exists thanks to Lemma 5.4.10. Next, let us choose $N_1 \geq 1$ such that if $n \geq N_1$, then $H(\mathcal{U}_s \mid \mathcal{P}_n^{t_0}) < \frac{\varepsilon}{3} n$.

In order to simplify our notation, let us denote $\mathcal{N}_n = \mathcal{N}_n(t_0, R)$ and $\beta_n = \beta_n(t_0, R, L)$. Recall the definition of $\Phi_n^{\text{in}}(t_0, R)$ as the the lamp configuration inside the coarse neighborhood (Definition 5.3.12).

Note that the value of φ_s at an element $b \in \mathcal{N}_n$ is completely determined by the lamp increments that affect position b, and this information is completely contained in β_n together with Δ_s. This implies that $H_{A^B}\left(\Phi_n^{\text{in}}(t_0,R) \middle| \mathcal{P}_n^{t_0} \vee \beta_n \vee \Delta_s\right) = 0$.

With the above, we can conclude that for every $n \geq N_1$, it holds that

$$\begin{aligned} H_{A^B}\left(\Phi_n^{\text{in}}(t_0,R) \middle| \mathcal{P}_n^{t_0} \vee \beta_n\right) \leq{}& H_{A^B}\left(\Phi_n^{\text{in}}(t_0,R) \middle| \mathcal{P}_n^{t_0} \vee \beta_n \vee \Delta_s\right) + H\left(\Delta_s \middle| \mathcal{P}_n^{t_0} \vee \mathcal{U}_s \vee \mathcal{V}_s\right) + \\ &+ H\left(\mathcal{V}_s\right) + H\left(\mathcal{U}_s \middle| \mathcal{P}_n^{t_0}\right) \\ \leq{}& 0 + \frac{\varepsilon}{3} n + K_4 + \frac{\varepsilon}{3} n + \frac{\varepsilon}{3} n = \varepsilon n + K_4. \end{aligned}$$

□

5.5 Proofs of the main theorems

In this section we present the proofs of Theorem 5.1.3 and Theorem 5.1.6. Both proofs have a similar structure, and the main difference between them is that for Theorem 5.1.6 we need to show that the additional hypothesis of a finite first moment implies that the coarse trajectory on the base group has low entropy, when conditioned on the limit lamp configuration. This fact relies on Proposition 5.5.2, which is proved in the next subsection.

The following proposition will be used in both proofs, and it states that the entropy of the position of the μ-random walk at time n is small, when conditioned on the t_0-coarse trajectory and the increments that occurred during the (R, L)-bad intervals. This is the combination of the main results of Sections 5.3 and 5.4, where we estimated the entropy of the lamp configuration at instant n outside of the (t_0, R)-coarse neighborhood and inside of the (t_0, R)-coarse neighborhood, respectively.

Proposition 5.5.1. *For any $\varepsilon > 0$, there exists $T \geq 1$ such that the following holds. Consider finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$, and $t_0 \geq T$. Then there exist $N \geq 1$ and $K \geq 0$ such that for every $n \geq N$,*

$$H_{A^B}\left(\varphi_n, X_n \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) < \varepsilon n + K.$$

Proof. Indeed, consider $\varepsilon > 0$ and $T \geq 1$ given by Proposition 5.4.1. Then for every choice of R, L and t_0 as above, there are $N \geq 1$ and $K \geq 0$ such that Equation (5.1) holds for every $n \geq N$. Let us write $\beta_n = \beta_n(t_0, R, L)$ and $\mathcal{N}_n = \mathcal{N}_n(t_0, R)$ in order to have less notation.

Using Lemma 5.3.10, Lemma 5.3.13 and the fact that $H(X_s \mid \mathcal{P}_n^{t_0}) = 0$, we have that

$$\begin{aligned}
H_{A^B}(\varphi_n, X_n \mid \mathcal{P}_n^{t_0} \vee \beta_n) &\leq H_{A^B}(\varphi_s \mid \mathcal{P}_n^{t_0} \vee \beta_n) + H_{A^B}(\varphi_n, X_n \mid \mathcal{P}_n^{t_0} \vee \beta_n \vee \varphi_s) \\
&= H_{A^B}(\varphi_s \mid \mathcal{P}_n^{t_0} \vee \beta_n) + 0 \\
&\leq H_{A^B}(\varphi_s|_{B \setminus \mathcal{N}_n} \mid \mathcal{P}_n^{t_0} \vee \beta_n) + H_{A^B}(\varphi_s|_{\mathcal{N}_n} \mid \mathcal{P}_n^{t_0} \vee \beta_n) \\
&\leq 0 + \varepsilon n + K.
\end{aligned}$$

□

Now we will prove Theorem 5.1.3. The proof of Theorem 5.1.6 is given below in Subsection 5.5.1.

Proof of Theorem 5.1.3. Thanks to Theorem 5.2.4, it suffices to prove that

$$\lim_{n \to \infty} \frac{H_{A^B \times \partial B}(\varphi_n, X_n)}{n} = 0.$$

Let $\varepsilon > 0$. Let us use Lemma 5.3.4 to find $T \geq 1$ such that for every $t_0 > T$, it holds that for every $n \geq 1$, $H_{\partial B}(\mathcal{P}_n^{t_0}) < \frac{\varepsilon}{3} n$.

Thanks to Proposition 5.5.1 we can find $T \geq 1$ large enough such that in addition to the conditions of the previous paragraph, the following holds for every $t_0 > T$. For any choice of finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$, there exist $N \geq 1$ and $K \geq 0$ such that for every $n \geq N$:

$$H_{A^B}\left(\varphi_n, X_n \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) < \frac{\varepsilon}{3}n + K.$$

Additionally, by virtue of Lemma 5.3.9 there exist R and L as above and $K_1 \geq 0$, such that

$$H\left(\beta_n(t_0, R, L)\right) < \frac{\varepsilon}{3}n + K_1 \text{ for every } n \geq 1.$$

Consider the associated values of N and K given by Proposition 5.5.1. Then for every $n \geq N$, we can use Lemma 5.2.7 to obtain

$$\begin{aligned} H_{A^B\times\partial B}\left(\varphi_n, X_n\right) &\leq H_{A^B\times\partial B}\left(\varphi_n, X_n \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) + H_{A^B\times\partial B}(\mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)) \\ &\leq H_{A^B\times\partial B}\left(\varphi_n, X_n \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) + H_{A^B\times\partial B}\left(\beta_n(t_0, R, L)\right) + \\ &\qquad + H_{A^B\times\partial B}(\mathcal{P}_n^{t_0}). \end{aligned}$$

Next, using Lemma 5.2.5 we have the following upper bounds:

$$H_{A^B\times\partial B}(\mathcal{P}_n^{t_0}) \leq H_{\partial B}(\mathcal{P}_n^{t_0}), \text{ and } H_{A^B\times\partial B}\left(\beta_n(t_0, R, L)\right) \leq H\left(\beta_n(t_0, R, L)\right).$$

Combining this with the above, we see that

$$\begin{aligned} H_{A^B\times\partial B}\left(\varphi_n, X_n\right) &\leq H_{A^B\times\partial B}\left(\varphi_n, X_n \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) + H\left(\beta_n(t_0, R, L)\right) + H_{\partial B}(\mathcal{P}_n^{t_0}). \\ &\leq \frac{\varepsilon}{3}n + K + \frac{\varepsilon}{3}n + K_1 + \frac{\varepsilon}{3}n \\ &= \varepsilon n + K + K_1. \end{aligned}$$

We conclude that for an arbitrary $\varepsilon > 0$, we have $\limsup_{n\to\infty} \frac{H_{A^B\times\partial B}(\varphi_n, X_n)}{n} \leq \varepsilon$, and hence

$$\lim_{n\to\infty} \frac{H_{A^B\times\partial B}\left(\varphi_n, X_n\right)}{n} = 0.$$

□

Proof of Corollary 5.1.5. It is proved in [Kaimanovich, 1991, Theorem 3.3] that if μ is a probability measure on $A \wr \mathbb{Z}^d$ such that its projection to $\mathbb{Z}^d$ defines a transient random walk, then the lamp configurations stabilize along almost every trajectory to a limit lamp configuration (this holds more generally for any probability measure on $A \wr B$ with a finite first moment that induces a transient random walk on the base group B [Erschler, 2011, Lemma 1.1]). Additionally, it is a well-known fact that probability measures with a finite first moment have finite entropy [Derriennic, 1986] (see also [Kaimanovich, 2001, Lemma 2.2.2]). The hypotheses of Theorem 5.1.3 are thus verified. Since the Poisson boundary of the induced random walk on $\mathbb{Z}^d$ is trivial [Blackwell, 1955] (see also [DoobSnellWilliamson, 1960] and [ChoquetDeny, 1960]), we obtain the desired result. □

5.5.1 Proof of Theorem 5.1.6

The objective of this subsection is to prove Theorem 5.1.6. The main difference with the proof of Theorem 5.1.3 is that Lemma 5.3.4 will be replaced by the following proposition. Intuitively, it states that, with the additional assumption of a finite first moment, the t_0-coarse trajectory has small entropy when conditioned on the boundary of limit lamp configurations.

Proposition 5.5.2. *Let μ be a probability measure on $A \wr B$ with $H(\mu) < \infty$ and such that $\langle \mathrm{supp}(\mu)\rangle_+$ contains two distinct elements with the same projection to B. Suppose that μ induces a transient random walk on B. Then for every $\varepsilon > 0$, there exists $T \geq 1$ such that for every $t_0 \geq T$,*

$$H_{A^B}(\mathcal{P}_n^{t_0}) < \varepsilon n, \text{ for every } n \geq 1.$$

Let us first explain how Theorem 5.1.6 follows from Proposition 5.5.2, which is proved afterward.

Proof of Theorem 5.1.6. We will show that

$$\lim_{n\to\infty} \frac{H_{A^B}(\varphi_n, X_n)}{n} = 0,$$

which thanks to Theorem 5.2.4 implies that $(A^B, \nu_{\mathcal{L}})$ is the Poisson boundary of $(A \wr B, \mu)$.

Recall that we suppose that μ has a finite first moment, and that lamp configurations stabilize almost surely. Thanks to Lemma 5.3.2, the latter hypothesis implies that the induced random walk on B is transient. We can thus use Proposition 5.5.2 to find for every $\varepsilon > 0$, a constant $T \geq 1$ such that for every $t_0 > T$, it holds that for every $n \geq 1$, $H_{A^B}(\mathcal{P}_n^{t_0}) < \frac{\varepsilon}{3} n$.

The rest of the proof is analogous to that of Theorem 5.1.3. We repeat the argument for the convenience of the reader.

Using Proposition 5.5.1 we can find $T \geq 1$ large enough such that in addition to the condition of the previous paragraph, the following holds for every $t_0 > T$. For any choice of finite subsets $R \subseteq B$ and $L \subseteq A$ with $e_B \in R$ and $e_A \in L$, there exist $N \geq 1$ and $K \geq 0$ such that for every $n \geq N$:

$$H_{A^B}\left(\varphi_n, X_n \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) < \frac{\varepsilon}{3} n + K.$$

Additionally, by virtue of Lemma 5.3.9 there exist R and L as above and $K_1 \geq 0$, such that

$$H(\beta_n(t_0, R, L)) < \frac{\varepsilon}{3} n + K_1 \text{ for every } n \geq 1.$$

Consider the associated values of N and K given by Proposition 5.5.1.

Then for every $t_0 > T$ and $n \geq N$, it follows from Lemma 5.2.7 that

$$H_{A^B}(\varphi_n, X_n) \leq H_{A^B}\left(\varphi_n, X_n \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) + H_{A^B}(\mathcal{P}_n^{t_0}) + H_{A^B}(\beta_n(t_0, R, L)).$$

Furthermore, we see from Lemma 5.2.5 that $H_{A^B}(\beta_n(t_0, R, L)) \leq H(\beta_n(t_0, R, L))$. Replacing above, we get

$$\begin{aligned} H_{A^B}(\varphi_n, X_n) &\le H_{A^B}\left(\varphi_n, X_n \mid \mathcal{P}_n^{t_0} \vee \beta_n(t_0, R, L)\right) + H_{A^B}(\mathcal{P}_n^{t_0}) + H(\beta_n(t_0, R, L)) \\ &\le \frac{\varepsilon}{3}n + K + \frac{\varepsilon}{3}n + \frac{\varepsilon}{3}n + K_1 \\ &= \varepsilon n + K + K_1. \end{aligned}$$

With this, we obtain that $\limsup_{n\to\infty} \frac{H_{A^B}(\varphi_n, X_n)}{n} \le \varepsilon$ for every $\varepsilon > 0$, and hence that

$$\lim_{n\to\infty} \frac{H_{A^B}(\varphi_n, X_n)}{n} = 0.$$

□

In what remains of the paper we prove Proposition 5.5.2.

Let us first recall the relation between the asymptotic entropy, the growth of the group and the speed of a random walk driven by a step distribution with a finite first moment.

Let G be a finitely generated group and let $|\cdot|$ be the word length with respect to some finite generating set of G. Let μ be a probability measure on G with a finite first moment, and denote by $\{w_n\}_{n\ge 0}$ the μ-random walk on G. It is a consequence of Kingman's subadditive ergodic theorem that the limit

$$\ell := \lim_{n\to\infty} \frac{\mathbb{E}(|w_n|)}{n} \tag{5.4}$$

exists, and that for $\mathbb{P}$-almost every trajectory of the μ-random walk one has $\ell = \lim_{n\to\infty} \frac{|w_n|}{n}$. The value of ℓ is called the *speed* of the random walk (G, μ). Additionally, let us denote $v := \lim_{n\to\infty} \frac{1}{n} \log\left(|\{g \in G \mid |g| \le n\}|\right)$ the *exponential growth rate* of G. Recall also Definition 5.1.2 for the asymptotic entropy of (G, μ).

The so-called "fundamental inequality" of Guivarc'h states that $h_\mu \le \ell v$ [Guivarch, 1980, Proposition 2]. In particular if the μ-random walk on G has a non-trivial Poisson boundary, it follows from the entropy criterion that $h_\mu > 0$, and hence that $\ell > 0$. This will be used below.

In order to prove Proposition 5.5.2, we would like to show that the Poisson boundary of the μ_B-random walk on B is completely described by the stabilization of the limit lamp configuration. However, we cannot directly apply the conditional entropy criterion. Indeed, this space is not a μ_B-boundary, so applying the usual results is not correct. Nonetheless, we can modify the definitions and follow the same ideas of Kaimanovich's conditional entropy criterion to prove our result. The following is similar to the exposition in [Kaimanovich, 2000, Section 4].

Recall our notation from Subsection 5.3.1 for the μ-random walk $\{(\varphi_n, X_n)\}_{n\ge 0}$, so that $(\varphi_n, X_n) = (f_1, Y_1)\cdots(f_n, Y_n)$, $n \ge 0$, with $\{(f_i, Y_i)\}_{i\ge 1}$ a sequence of independent identically distributed random variables with law μ. Recall also the definitions associated with entropy, partitions and conditional entropy of Subsection 5.2.3.

We define

$$h\left(B, \mu_B \mid A^B\right) := \lim_{n\to\infty} \frac{H_{A^B}(X_n)}{n},$$

which is well-defined since the sequence $\{H_{A^B}(X_n)\}_{n\ge 0}$ is subadditive.

For every $\varphi_\infty \in A^B$ denote by $p_n^{\varphi_\infty}$, $n \geq 0$, the one-dimensional distributions of the measure $\mathbb{P}^{\varphi_\infty}$. It is a consequence of Kingman's subadditive ergodic theorem that for $\mathbb{P}^{\varphi_\infty}$-almost every trajectory $\{(\varphi_n, X_n)\}_{n\geq 0}$, the limit

$$h\left(B, \mu_B \mid \varphi_\infty\right) := -\lim_{n\to\infty} \frac{\log(p_n^{\varphi_\infty}(X_n))}{n}$$

is well defined and a constant. Furthermore, we have

$$h\left(B, \mu_B \mid \varphi_\infty\right) = \lim_{n\to\infty} \frac{\mathbb{E}\Big(- \log(p_n^{\varphi_\infty}(X_n))\Big)}{n} = \lim_{n\to\infty} \frac{H_{\varphi_\infty}(X_n)}{n},$$

so that after integrating we have

$$h\left(B, \mu_B \mid A^B\right) = \int_{A^B} h\left(B, \mu_B \mid \varphi_\infty\right) \, d\nu_{\mathcal{L}}(\varphi_\infty). \tag{5.5}$$

With this, Proposition 5.5.2 will follow from proving that $h\left(B, \mu_B \mid A^B\right) = 0$ in Lemma 5.5.4 below.

If the induced random walk on B is Liouville, then it follows from the usual entropy criterion that $h\left(B, \mu_B\right) = 0$, which implies the above. Hence, we will suppose from now on that the induced random walk on B is not Liouville. Denote by ℓ the speed of the μ_B-random walk on B. It follows from the entropy criterion together with the fundamental inequality of Guivarc'h that $\ell > 0$.

Lemma 5.5.3. *Let μ be a probability measure on $A \wr B$ with a finite first moment and such that $\langle \mathrm{supp}(\mu)\rangle_+$ contains two distinct elements with the same projection to B. Suppose that the random walk induced by μ on B is not Liouville. Then, for $\nu_{\mathcal{L}}$-almost-every $\varphi_\infty \in A^B$ there is a family of finite subsets $Q_n \subseteq B$, $n \geq 1$, such that*

1. $\limsup_{n\to\infty} \mathbb{P}^{\varphi_\infty}(X_n \in Q_n) > 0$, *and*
2. $\lim_{n\to\infty} \frac{1}{n} \log |Q_n| = 0$.

Proof. Let S be an arbitrary finite generating set of B and let us denote by $d_S(\cdot,\cdot)$ and $|\cdot|_S$ the associated word metric and word length. Given an element $b \in B$ and a non-empty subset $F \subseteq B$, we denote $d_S(b, F) := \min\{d_S(b, f) \mid f \in F\}$.

We first note that the speed ℓ of the μ_B-random walk, as defined in Equation (5.4), satisfies that $\mathbb{P}$-almost surely we have that for every n large enough,

$$(\ell - \varepsilon)n \leq |X_n|_S \leq (\ell + \varepsilon)n. \tag{5.6}$$

Additionally, the finite first moment hypothesis implies that $\mathbb{P}$-almost surely for all n large enough,

the lamp increment f_n only modifies lamps at a d_S-distance at most εn. (5.7)

Since μ has a finite first moment and is adapted, it is a consequence of Kingman's subadditive ergodic theorem that there exists $C \geq 0$ such that

$$\lim_{n\to\infty} \frac{|\text{supp}(\varphi_n)|}{n} = C.$$

This is proved in [Erschler, 2011, Lemma 2.1] (see also [Gilch, 2008, Theorem 3.1]). Then, it holds that that $\mathbb{P}$-almost surely for every n large enough, we have

$$(C-\varepsilon)n \leq |\text{supp}(\varphi_n)| \leq (C+\varepsilon)n \tag{5.8}$$

Let us choose $N \geq 1$ large enough so that Equations (5.6), (5.7) and (5.8) hold for all $n \geq N$.

We will now argue that with positive probability, the position of the random walk X_n in the base group is close to a position $b \in B$ on which the limit lamp configuration is turned on: $\varphi_\infty(b) \neq e_A$.

Since we are assuming that $\langle \text{supp}(\mu)\rangle_+$ contains two distinct elements with the same projection to B, there exist $k \geq 1$ and $g, g' \in \text{supp}(\mu^{*k})$ with the same projection to B, and with lamp configurations $f, f' \in \bigoplus_B A$ such that $f \neq f'$. Let $D > 0$ be large enough such that $\text{supp}(f)$ and $\text{supp}(f')$ are contained in the ball of radius D centered at e_B.

Now let us look at sample paths $w = (w_1, w_2, \dots) \in (A \wr B)^\infty$, and condition on all of the increments except for those that occur at instants $n+1, n+2, \dots, n+k$, and such that the product of the increments $g_{n+1}g_{n+2}\cdots g_{n+k}$ is either g or g'. For such sample paths, the values of the limit lamp configuration φ_∞ are always the same except possibly at positions within the ball of radius D of X_n. Since the lamp configurations of g and g' are distinct, then in at least one of the two limit lamp configurations there will be $b \in B$ with $d_S(X_n, b) \leq D$ and with $\varphi_\infty(b) \neq e_A$. This implies that

$$\mathbb{P}^{\varphi_\infty}\Big(d_S(X_n, \text{supp}(\varphi_\infty)) \leq D\Big) \geq \min\{\mu^{*k}(g), \mu^{*k}(g')\} > 0,$$

for all $n \geq 1$, and hence that

$$\limsup_{n\to\infty} \mathbb{P}^{\varphi_\infty}\Big(d_S(X_n, \text{supp}(\varphi_\infty)) < D\Big) > 0. \tag{5.9}$$

Let us define for every $n \geq 1$ the finite subset $Q_n \subseteq B$ as the set of elements $b \in B$ such that $(\ell-\varepsilon)n \leq |b|_S \leq (\ell+\varepsilon)n$, and $d_S(b, \text{supp}(\varphi_\infty)) < D$. Item (5.6) above together with Equation (5.9) imply that $\limsup_{n\to\infty} \mathbb{P}^{\varphi_\infty}(X_n \in Q_n) > 0$. It remains to estimate the size of Q_n for n large enough.

Let us denote the ball of radius r in B by $\text{Ball}(r) \coloneqq \{b \in B \mid |b|_S \leq r\}$. Note that any element $b \in Q_n$ is at distance at most D of some $x \in \text{supp}(\varphi_\infty)$, such that $(\ell-\varepsilon)n - D \leq |x|_S \leq (\ell+\varepsilon)n + D$. Thus, we have that

$$|Q_n| \leq \Big|\text{supp}(\varphi_\infty) \cap \Big(\text{Ball}((\ell+\varepsilon)n + D) \backslash \text{Ball}(\ell-\varepsilon)n - D)\Big)\Big|. \tag{5.10}$$

Recall that for large enough n, Equations (5.6), (5.7) and (5.8) above hold. Thanks to this, the last instant k_1 that some lamp inside $\text{Ball}((\ell+\varepsilon)n+D)$ is modified satisfies $(\ell-\varepsilon)k_1 - \varepsilon n \leq$

$(\ell+\varepsilon)n + D$, so that
$$k_1 \leq \frac{\ell+2\varepsilon}{\ell-\varepsilon}n + \frac{D}{\ell-\varepsilon}.$$

Similarly, any modification to a lamp outside of $\mathrm{Ball}(\ell-\varepsilon)n - D)$ can only occur after the time instant k_2 that satisfies $k_2(\ell+\varepsilon)+\varepsilon n \geq (\ell-\varepsilon)n - D$, so that
$$k_2 \geq \frac{\ell-2\varepsilon}{\ell+\varepsilon}n - \frac{D}{\ell+\varepsilon}.$$

We conclude from the above together with Item (5.8) that for n large enough, the set $\mathrm{supp}(\varphi_\infty)$ contains at most $(C+\varepsilon)\left(\frac{\ell+2\varepsilon}{\ell-\varepsilon}n + \frac{D}{\ell-\varepsilon}\right)$ elements inside $\mathrm{Ball}((\ell+\varepsilon)n + D)$, and at least $(C-\varepsilon)\left(\frac{\ell-2\varepsilon}{\ell+\varepsilon}n - \frac{D}{\ell+\varepsilon}\right)$ inside $\mathrm{Ball}((\ell-\varepsilon)n - D)$.

Now we can use Equation (5.10) to obtain
$$\begin{aligned}|Q_n| &\leq (C+\varepsilon)\left(\frac{\ell+2\varepsilon}{\ell-\varepsilon}n + \frac{D}{\ell-\varepsilon}\right) - (C-\varepsilon)\left(\frac{\ell-2\varepsilon}{\ell+\varepsilon}n - \frac{D}{\ell+\varepsilon}\right)\\ &= C_1 n + C_2,\end{aligned}$$

for some appropriate constants $C_1, C_2 > 0$ whose explicit value is not relevant. From this we obtain that $\lim_{n\to\infty}\frac{1}{n}\log|Q_n| = 0$, hence finishing the proof. □

Lemma 5.5.4. *We have* $h\left(B, \mu_B \mid A^B\right) = 0$.

Proof. Looking for a contradiction, let us suppose that $h\left(B, \mu_B \mid A^B\right) > 0$. Thanks to Equation (5.5), it must hold that for φ_∞ in a measurable subset $T \subseteq A^B$ of positive $\nu_{\mathcal{L}}$-measure we have $h := h(B, \mu_B \mid \varphi_\infty) > 0$.

For $\varphi_\infty \in T$ and $n \geq 1$, let us define the set
$$E_n := \left\{b \in B \mid p_n^{\varphi_\infty}(b) < e^{-nh/2}\right\}.$$

Since for $\mathbb{P}^{\varphi_\infty}$-almost-every trajectory $\{(\varphi_n, X_n)\}_{n\geq 1}$ we have $\lim_{n\to\infty}\frac{1}{n}\log p_n^{\varphi_\infty}(X_n) = -h$, then it holds that $\lim_{n\to\infty}\mathbb{P}^{\varphi_\infty}(X_n \in E_n) = 1$ for every $\varphi_\infty \in T$.

By virtue of Lemma 5.5.3 we can find finite subsets $Q_n \subseteq B$, $n \geq 1$, such that
$$\lim_{n\to\infty}\frac{1}{n}\log|Q_n| = 0 \text{ and } \limsup_{n\to\infty}\mathbb{P}^{\varphi_\infty}(X_n \in Q_n) > 0.$$

Then, we see that
$$\begin{aligned}\mathbb{P}^{\varphi_\infty}(X_n \in E_n \cap Q_n) &= \sum_{b\in E_n\cap Q_n}\mathbb{P}^{\varphi_\infty}(X_n = b)\\ &\leq |Q_n|e^{-nh/2}.\end{aligned}$$

However, we have that $\lim_{n\to\infty}|Q_n|e^{-nh/2} = 0$. Indeed,
$$\log(|Q_n|e^{-nh/2}) = n\left(\frac{1}{n}\log|Q_n| - \frac{h}{2}\right) \xrightarrow[n\to\infty]{} -\infty.$$

This implies that $\lim_{n\to\infty} \mathbb{P}^{\varphi_\infty}(X_n \in E_n \cap Q_n) = 0$. Furthermore, since we have that $\lim_{n\to\infty} \mathbb{P}^{\varphi_\infty}(X_n \in E_n) = 1$, it must hold that $\lim_{n\to\infty} \mathbb{P}^{\varphi_\infty}(X_n \in Q_n) = 0$. This contradicts our choice of the sets Q_n. □

With this, we can now prove Proposition 5.5.2.

Proof of Proposition 5.5.2. Thanks to Lemma 5.5.4, we have that

$$\lim_{n\to\infty} \frac{H_{A^B}(X_n)}{n} = 0.$$

This implies that for every $\varepsilon > 0$, there exists $T \geq 1$ such that $H_{A^B}(X_{t_0}) < \varepsilon t_0$ for every $t_0 \geq T$. Afterward, the proof follows analogously to the proof of Lemma 5.3.4. Indeed, let $n \geq 1$, $t_0 \geq T$, and denote $s = \lfloor n/t_0 \rfloor$. Then

$$H_{A^B}(X_{t_0}, X_{2t_0}, \dots, X_{st_0}) \leq \sum_{j=0}^{s-1} H_{A^B}\left(X_{(j+1)t_0} \mid X_{jt_0}\right) = sH_{A^B}(X_{t_0}) < s\varepsilon t_0 \leq \varepsilon n.$$

□

We remark that the assumption that $\langle \mathrm{supp}(\mu) \rangle_+$ contains two distinct elements with the same projection to B in Theorem 5.1.6 and Proposition 5.5.2 is necessary, as the following example shows.

Example 5.5.5. Consider the wreath product $\mathbb{Z}/2\mathbb{Z} \wr F_2$, where F_2 is the free group of rank 2. We denote $\mathbb{Z}/2\mathbb{Z} = \{[0]_2, [1]_2\}$, where $[0]_2 \in \mathbb{Z}/2\mathbb{Z}$ is the identity element. Denote by $\delta_0 : F_2 \to \mathbb{Z}/2\mathbb{Z}$ the lamp configuration defined by $\delta_0(e_{F_2}) = [1]_2$ and $\delta_0(x) = [0]_2$ for $x \in F_2 \backslash \{e_{F_2}\}$. Let us also denote by $\{a, b\}$ a free generating set of F_2.

Let μ be the probability measure on $\mathbb{Z}/2\mathbb{Z} \wr F_2$ defined by $\mu(\delta_0 x \delta_0) = 1/4$ for each $x \in \{a, a^{-1}, b, b^{-1}\}$. Then the projection of the μ-random walk on $\mathbb{Z}/2\mathbb{Z} \wr F_2$ to F_2 is a simple random walk, and in particular this implies that the Poisson boundary is non-trivial. However, note that the limit lamp configuration φ_∞ along any sample path satisfies that $\varphi_\infty(x) = [0]_2$ for all $x \in F_2 \backslash \{e_{F_2}\}$. The μ-boundary of infinite lamp configurations is then trivial, and hence it is not the Poisson boundary.

5.6 Groups of the form $F_d/[N, N]$ and free solvable groups

5.6.1 The Magnus embedding

Let F_d be a free group of rank $d \geq 1$, and consider a normal subgroup $N \lhd F_d$. We now recall the Magnus embedding of $F_d/[N, N]$ into the wreath product $\mathbb{Z}^d \wr F_d/N$. We follow the expositions of [MyasnikovRomankovUshakovVershik, 2010, Subsections 2.2 – 2.5] and [Saloff-CosteZheng, 2015, Section 2], which are our main references for this subsection.

We first introduce the notation that we will use. We denote by $\pi : F_d \to F_d/N$, $\widetilde{\pi} : F_d/[N, N] \to F_d/N$ and $\theta : F_d \to F_d/[N, N]$ the respective quotient maps. Note that $\pi = \widetilde{\pi} \circ \theta$.

We denote by $\mathbb{Z}(F_d/N)$ and $\mathbb{Z}(F_d)$ the group rings with integer coefficients of F_d/N and F_d, respectively. Let us also denote by $\pi : \mathbb{Z}(F_d) \to \mathbb{Z}(F_d/N)$ the linear extension of the quotient map $\pi : F_d \to F_d/N$.

Let T be a free left $\mathbb{Z}(F_d/N)$-module of rank d with basis $\{t_1, \dots, t_d\}$, and consider the group of matrices

$$M(F_d/N) := \left\{ \begin{pmatrix} g & t \\ 0 & 1 \end{pmatrix} \mid g \in F_d/N \text{ and } t \in T \right\},$$

where the group operation is matrix multiplication. One can verify with a direct computation that T is isomorphic to the direct sum $\bigoplus_{F_d/N} \mathbb{Z}^d$, and that $M(F_d/N)$ is isomorphic to $\mathbb{Z}^d \wr F_d/N$.

The following theorem describes the Magnus embedding of $F_d/[N,N]$ into $\mathbb{Z}^d \wr F_d/N$.

Theorem 5.6.1 ([Magnus, 1939]). *Let F_d be a free group of rank $d \geq 1$, and consider $X = \{x_1, \dots, x_d\}$ a free basis of F_d. Let $N \lhd F_d$ be a normal subgroup and denote by $\pi : F_d \to F_d/N$ the quotient map. Then the homomorphism $\phi : F_d \to M(F_d/N)$ defined by*

$$\phi(x_i) := \begin{pmatrix} \pi(x_i) & t_i \\ 0 & 1 \end{pmatrix}, \qquad \text{for } i = 1, \dots, d,$$

satisfies $\ker(\phi) = [N,N]$. *Hence, ϕ induces a monomorphism $\widetilde{\phi} : F_d/[N,N] \to M(F_d/N)$.*

In other words, the Magnus embedding $\widetilde{\phi} : F_d/[N,N] \to M(F_d/N) \cong \mathbb{Z}^d \wr F_d/N$ can be written as $\widetilde{\phi}(g) = (\varphi(g), \widetilde{\pi}(g))$ for each $g \in F_d/[N,N]$, where $\varphi(g) \in \bigoplus_{F_d/N} \mathbb{Z}^d$ and $\widetilde{\pi} : F_d/[N,N] \to F_d/N$ is the quotient map. In order to describe more precisely the map $\varphi : F_d/[N,N] \to \bigoplus_{F_d/N} \mathbb{Z}^d$, we need to introduce Fox derivatives.

Consider the group ring $\mathbb{Z}(F_d)$, and let us denote by $\varepsilon : \mathbb{Z}(F_d) \to \mathbb{Z}$ the extension of the trivial group homomorphism $F_d \to 1$. A map $D : \mathbb{Z}(F_d) \to \mathbb{Z}(F_d)$ is called a *left derivation* if it satisfies:

— $D(u+v) = D(u) + D(v)$, for every $u, v \in \mathbb{Z}(F_d)$, and

— $D(uv) = D(u)\varepsilon(v) + uD(v)$, for every $u, v \in \mathbb{Z}(F_d)$.

The following two theorems are due to Fox [ChenFoxLyndon, 1958; Fox, 1953; Fox, 1954; Fox, 1956; Fox, 1960].

Theorem 5.6.2 (Fox). *Let F_d be a free group of rank $d \geq 1$, and consider $X = \{x_1, \dots, x_d\}$ a free basis of F_d. Then for each $i = 1, \dots, d$ there is a unique left derivation $\partial_i : \mathbb{Z}(F_d) \to \mathbb{Z}(F_d)$ such that*

$$\partial_i(x_j) = \begin{cases} 1, & \text{if } i = j, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, let $N \lhd F_d$ a normal subgroup and denote by $\pi : F_d \to F_d/N$ the quotient map. Then for every $u \in F_d$ it holds that

$$u \in [N,N] \iff \pi(\partial_i u) = 0 \text{ for every } i = 1, \dots, d.$$

Theorem 5.6.3 (Fox)**.** *Let F_d be a free group of rank $d \geq 1$ and let $N \lhd F_d$ a normal subgroup. Denote by $\pi : F_d \to F_d/N$ and by $\theta : F_d \to F_d/[N,N]$ the associated quotient maps. Then the Magnus embedding $\widetilde{\phi} : F_d/[N,N] \to M(F_d/N)$ can be expressed as*

$$\widetilde{\phi}(g) = \begin{pmatrix} \widetilde{\pi}(g) & \sum_{i=1}^{d} \pi(\partial_i u) \cdot t_i \\ 0 & 1 \end{pmatrix}, \qquad \textit{for } g \in F_d/[N,N],$$

where $u \in F_d$ is any element such that $\theta(u) = g$.

5.6.2 Flows

We now explain how the elements of the group $F_d/[N,N]$ can be expressed in terms of flows in the Cayley graph of F_d/N. We first need to introduce the notation and definitions associated with Cayley graphs and flows.

Let F_d be the free group of rank d and let $N \lhd F_d$ be a normal subgroup. By slightly abusing notation, we will denote by $X = \{x_1, \ldots, x_d\}$ a free generating set of F_d, as well as its image on F_d/N via the canonical quotient map $\pi : F_d \to F_d/N$.

The *directed and labeled Cayley graph* $\mathrm{Cay}(F_d/N)$ is the graph with vertex set $V = F_d/N$, and with edges

$$\mathrm{Edges}(F_d/N) \coloneqq \{(g_1, g_2, x) \in V \times V \times \{x_1, \ldots, x_d\} \mid g_2 = g_1 x\}.$$

Hence, each edge is of the form $\mathfrak{e} \coloneqq (g, gx_i, x_i)$ for some $g \in F_d/N$ and $i \in \{1, \ldots, d\}$. We call $g = o(\mathfrak{e})$ the *origin* of $\mathfrak{e}$, $gx_i = t(\mathfrak{e})$ the terminus of $\mathfrak{e}$, and x_i the label of $\mathfrak{e}$.

Definition 5.6.4.

1. Let $\mathfrak{f} : \mathrm{Edges}(F_d/N) \to \mathbb{Z}$ be a function. For each $g \in F_d/N$ we define the *net flow* $\mathfrak{f}^*(g)$ of $\mathfrak{f}$ at g by

$$\mathfrak{f}^*(g) \coloneqq \sum_{o(\mathfrak{e})=g} f(\mathfrak{e}) - \sum_{t(\mathfrak{e})=g} f(\mathfrak{e}).$$

2. Let $s, t \in F_d/N$. A *flow* with source s and sink t is a function $\mathfrak{f} : \mathrm{Edges}(F_d/N) \to \mathbb{Z}$ such that $f^*(g) = 0$ for all $g \in F_d/N \backslash \{s, t\}$. If this condition is also satisfied for $g = s$ and $g = t$, we call the flow $\mathfrak{f}$ a *circulation.*
3. We say that a flow $\mathfrak{f}$ is *geometric* if the set $\{\mathfrak{e} \in \mathrm{Edges}(F_d/N) \mid \mathfrak{f}(\mathfrak{e}) \neq 0\}$ is finite (i.e. $\mathfrak{f}$ is finitely supported), and if $\mathfrak{f}$ is either a circulation or it is a flow with source s and sink t such that $\mathfrak{f}^*(s) = 1$ and $\mathfrak{f}^*(t) = -1$.

For each edge $\mathfrak{e} = (g, gx, x) \in \mathrm{Edges}(F_d/N)$, let us introduce its formal inverse $\mathfrak{e}^{-1} = (gx, g, x^{-1})$. Denote by $\mathrm{Edges}(F/N)^{\pm 1}$ the set of edges together with their formal inverses.

A *finite path p* on the Cayley graph $\mathrm{Cay}(F_d/N)$ is a sequence $p = (\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_k)$ of edges $\mathfrak{e}_j \in \mathrm{Edges}(F/N)^{\pm 1}$, $j = 1, \ldots, k$, such that $t(\mathfrak{e}_j) = o(\mathfrak{e}_{j+1})$ for each $j = 1, 2, \ldots, k-1$. The element $o(\mathfrak{e}_1)$ is called the *origin of p* and the element $t(\mathfrak{e}_k)$ is called the *terminus of p*. Note that each finite path defines a word in F_d, given by concatenating the labels of edges along the path, and reciprocally any word in F_d defines a unique path with origin at the identity element $e_{F_d/N}$.

Let p be a finite path on the Cayley graph $\mathrm{Cay}(F_d/N)$. Then p defines a geometric flow $\mathfrak{f}_p : \mathrm{Edges}(F/N) \to \mathbb{Z}$, given by

$$\begin{aligned}\mathfrak{f}_p(\mathfrak{e}) &:= \text{algebraic number of times that } \mathfrak{e} \text{ is crossed (positively or negatively) along } p\\ &= (\text{number of times that } \mathfrak{e} \text{ occurs in } p) - (\text{number of times that } \mathfrak{e}^{-1} \text{ occurs in } p),\end{aligned}$$

for each $\mathfrak{e} \in \mathrm{Edges}(F/N)$.

For every element $g \in F_d/[N,N]$, choose any word $u \in F_d$ which projects to g and define the flow associated with g by $\mathfrak{f}_g := \mathfrak{f}_u$. The following theorem guarantees that $\mathfrak{f}_g$ is well-defined and does not depend on the choice of the word u.

Theorem 5.6.5 ([MyasnikovRomankovUshakovVershik, 2010, Theorem 2.7])**.** *Let F_d be a free group of rank $d \geq 1$ and let $N \lhd F_d$ be a normal subgroup. Denote by $\theta : F_d \to F_d/[N,N]$ the associated quotient map. Then for any $u, v \in F_d$ it holds that $\theta(u) = \theta(v)$ if and only if $\mathfrak{f}_u = \mathfrak{f}_v$.*

Recall that we denote by π the canonical epimorphism $F_d \to F_d/N$ as well as its linear extension $\mathbb{Z}(F_d) \to \mathbb{Z}(F_d/N)$. The following result expresses, for each $i = 1, \dots, d$ and $u \in F_d$, the values $\pi(\partial_i u)$ in terms of the flow $\mathfrak{f}_u$ on the Cayley graph of F_d/N associated with u.

Lemma 5.6.6 ([MyasnikovRomankovUshakovVershik, 2010, Lemma 2.6])**.** *Let F_d be a free group of rank $d \geq 1$ with a free generating set $\{x_1, \dots, x_d\}$, and let $N \lhd F_d$ a normal subgroup. Denote by $\pi : \mathbb{Z}(F_d) \to \mathbb{Z}(F_d/N)$ the linear extension of the quotient map $F_d \to F_d/N$. Then for every $u \in F_d$ and any $i = 1, \dots, d$ we have*

$$\pi(\partial_i u) = \sum_{g \in F_d/N} \mathfrak{f}_u((g, gx_i, x_i))g.$$

Remark 5.6.7. Note that, for any element $g \in F_d/[N,N]$, the flow $\mathfrak{f}_g$ completely determines the projection $\widetilde{\pi}(g) \in F_d/N$. Indeed, if $\mathfrak{f}_g$ is a circulation, then $\widetilde{\pi}(g) = e_{F_d/N}$ is the identity element, and otherwise $\widetilde{\pi}(g)$ is the sink of the geometric flow $\mathfrak{f}_g$. This observation together with Lemma 5.6.6 and Theorem 5.6.3 imply that the element $g \in F_d/[N,N]$ is completely determined by the flow $\mathfrak{f}_g$.

5.6.3 Proofs of Corollaries 5.1.8 and 5.1.9

Now we consider random walks on the group $F_d/[N,N]$, for $N \lhd F_d$ a normal subgroup of the free group F_d of rank d.

In analogy with the stabilization of the lamp configuration on wreath products (Definition 5.3.1), we define the stabilization of the flows associated with elements of $F_d/[N,N]$.

Definition 5.6.8. Let μ be a probability measure on the group $F_d/[N,N]$. We say that the flows $\{\mathfrak{f}_{w_n}\}_{n\geq 0}$ associated with the μ-random walk $\{w_n\}_{n\geq 0}$ on $F_d/[N,N]$ *stabilize almost surely* if for every $\mathfrak{e} \in \mathrm{Edges}(F_d/N)$, there exists $N \geq 1$ such that $\mathfrak{f}_{w_n}(\mathfrak{e}) = \mathfrak{f}_{w_N}(\mathfrak{e})$ for every $n \geq N$.

The stabilization of flows on the group $F_d/[N,N]$ is directly related to the stabilization of the lamp configuration in the wreath product $\mathbb{Z}^d \wr F_d/N$ via the Magnus embedding, as we state in the following lemma.

Lemma 5.6.9. *Let μ be a probability measure on the group $F_d/[N,N]$, and denote by $\widetilde{\phi}$: $F_d/[N,N] \to \mathbb{Z}^d \wr F_d/N$ the Magnus embedding. Then the flows $\{\mathfrak{f}_{w_n}\}_{n\geq 0}$ associated with the μ-random walk $\{w_n\}_{n\geq 0}$ on $F_d/[N,N]$ stabilize almost surely if and only if the lamp configurations of $\left\{\widetilde{\phi}(w_n)\right\}_{n\geq 0}$ stabilize almost surely. In such a case, the edgewise limit $\mathfrak{f}_\infty : \mathrm{Edges}(F_d/N) \to \mathbb{Z}$ of the flows completely determines the pointwise limit $\varphi_\infty : F_d/N \to \mathbb{Z}^d$ of the lamp configurations and vice-versa.*

Proof. For each sample path $\{w_n\}_{n\geq 0} \in (F_d/[N,N])^\infty$, let us consider its image $\left\{\widetilde{\phi}(w_n)\right\}_{n\geq 0} \in (\mathbb{Z}^d \wr F_d/N)^\infty$ via the Magnus embedding. Theorem 5.6.3 states that we can write

$$\widetilde{\phi}(w_n) = \begin{pmatrix} \widetilde{\pi}(w_n) & \sum_{i=1}^d \pi(\partial_i u_n)\cdot t_i \\ 0 & 1 \end{pmatrix},$$

where $u_n \in F_d$ is any element such that $\theta(u_n) = w_n$. In addition, by virtue of Lemma 5.6.6, we can write

$$\pi(\partial_i u_n) = \sum_{g\in F_d/N} \mathfrak{f}_{w_n}((g, gx_i, x_i))g, \text{ for each } i = 1,\dots,d.$$

Let us denote by $\varphi_n : F_d/N \to \mathbb{Z}^d$ the lamp configuration of $\widetilde{\phi}(w_n)$. Then for each $g \in F_d/N$, the above implies that

$$\varphi_n(g) = \sum_{i=1}^d \mathfrak{f}_{w_n}((g, gx_i, x_i))t_i, \text{ for all } n \geq 1. \tag{5.11}$$

Equation (5.11) directly relies the value of the lamp configuration at an element $g \in F_d/[N,N]$ with the values of the flow at edges $\mathfrak{e} \in \mathrm{Edges}(F_d/N)$ with origin $o(\mathfrak{e}) = g$. Since Equation (5.11) consists of a finite sum, we conclude that the flows stabilize along a sample path if and only if the lamp configuration stabilizes along the image of the sample path via the Magnus embedding. In such a case, we see also from Equation (5.11), that the edgewise limit flow $\mathfrak{f}_\infty$ and the pointwise limit lamp configuration φ_∞ satisfy

$$\varphi_\infty(g) = \sum_{i=1}^d \mathfrak{f}_\infty((g, gx_i, x_i))t_i, \text{ for all } g \in F_d/N.$$

We conclude that φ_∞ completely determines $\mathfrak{f}_\infty$ and vice-versa. □

Suppose that the flows stabilize almost surely along sample paths of the μ-random walk on $F_d/[N,N]$. this implies that there is a shift-invariant map $\mathcal{F} : (F_d/[N,N])^\infty \to \mathbb{Z}^{\mathrm{Edges}(F_d/N)}$ that is measurable and well-defined for $\mathbb{P}$-a.e. sample path $\{w_n\}_{n\geq 0}$ of the μ-random walk on $F_d/[N,N]$, given by the edgewise limit of the flows: $\mathcal{F}((w_n)_{n\geq 0}) := \lim_{n\to\infty} \mathfrak{f}_{w_n}$. The hitting measure is $\nu_\mathcal{F} := \mathcal{F}_*\mathbb{P}$. Hence, the space $(\mathbb{Z}^{\mathrm{Edges}(F_d/N)}, \nu_\mathcal{F})$ is a μ-boundary. Let us now consider the group $B = F_d/N$ and the canonical projection $\widetilde{\pi} : F_d/[N,N] \to B$. Denote by $(\partial B, \nu_B)$ the Poisson boundary of the $\widetilde{\pi}_*\mu$-random walk on B, and denote by $\mathrm{bnd}_B : B^\infty \to \partial B$ the corresponding boundary map. Then the map

$$\begin{aligned}(F_d/[N,N])^\infty &\to \mathbb{Z}^{\mathrm{Edges}(F_d/N)} \times \partial B\\ w &\mapsto \left(\mathcal{F}(w), \mathrm{bnd}_B\left(\{\widetilde{\pi}(w_n)\}_{n\geq 0}\right)\right),\end{aligned}$$

pushes forward $\mathbb{P}$ to a μ-stationary measure ν on $\mathbb{Z}^{\mathrm{Edges}(F_d/N)} \times \partial B$. With this, the space $(\mathbb{Z}^{\mathrm{Edges}(F_d/N)} \times \partial B, \nu)$ is a μ-boundary of $F_d/[N,N]$.

Lemma 5.6.10. *Let μ be a probability measure on the group $F_d/[N,N]$, and denote by $\widetilde{\phi} : F_d/[N,N] \to \mathbb{Z}^d \wr F_d/N$ the Magnus embedding. Denote by $\{w_n\}_{n\geq 0}$ a sample path for the μ-random walk on $F_d/[N,N]$. Then we have*

$$H_{\mathbb{Z}^{\mathrm{Edges}(F_d/N)}\times\partial B}(w_n) = H_{(\mathbb{Z}^d)^B\times\partial B}\left(\widetilde{\phi}(w_n)\right), \text{ for every } n \geq 1, \text{ and} \tag{5.12}$$

$$H_{\mathbb{Z}^{\mathrm{Edges}(F_d/N)}}(w_n) = H_{(\mathbb{Z}^d)^B}\left(\widetilde{\phi}(w_n)\right), \text{ for every } n \geq 1, \tag{5.13}$$

where $B = F_d/N$.

Proof. This result follows from Lemma 5.6.9, which states that the edgewise limit flow is completely determined by the pointwise limit lamp configuration, as well as the other way around. To prove Equation (5.12), note that we have

$$\begin{aligned}H_{\mathbb{Z}^{\mathrm{Edges}(F_d/N)}\times\partial B}(w_n) &= \int_{\mathbb{Z}^{\mathrm{Edges}(F_d/N)}\times\partial B} H_{(\mathfrak{f}_\infty,\xi)}(w_n)\, d\nu(\mathfrak{f}_\infty,\xi)\\ &= \int_{(\mathbb{Z}^d)^B\times\partial B} H_{(\varphi_\infty,\xi)}(\widetilde{\phi}(w_n))\, d\nu(\varphi_\infty,\xi)\\ &= H_{(\mathbb{Z}^d)^B\times\partial B}\left(\widetilde{\phi}(w_n)\right).\end{aligned}$$

Equation (5.13) is proved in a completely analogous way, noting that Lemma 5.6.9 does not depend on what the Poisson boundary of the induced random walk on B is. □

We finish the paper with the proofs of Corollaries 5.1.8 and 5.1.9.

Proof of Corollary 5.1.8. Denote by $\widetilde{\phi} : F_d/[N,N] \to \mathbb{Z}^d \wr F_d/N$ the Magnus embedding. If μ is a probability measure with finite entropy, then $\widetilde{\phi}_*\mu$ also has finite entropy. Furthermore, if the flows stabilize almost surely along sample paths of the μ-random walk on $F_d/[N,N]$, then Lemma 5.6.9 implies that the lamp configuration stabilizes along sample paths of the $\widetilde{\phi}_*\mu$-random walk on $\mathbb{Z}^d \wr F_d/N$. Then, Theorem 5.1.3 implies that the Poisson boundary of $(\mathbb{Z}^d \wr F_d/N, \widetilde{\phi}_*\mu)$ is the space $(\mathbb{Z}^d)^B \times \partial B$ with the corresponding harmonic measure, where we denote $B \coloneqq F_d/N$. Then, the conditional entropy criterion (Theorem 5.2.4) implies that $\lim_{n\to\infty} \frac{H_{(\mathbb{Z}^d)^B\times\partial B}\left(\widetilde{\phi}(w_n)\right)}{n} = 0$, and we conclude from Equation (5.12) of Lemma 5.6.10, that

$$\lim_{n\to\infty} \frac{H_{\mathbb{Z}^{\mathrm{Edges}(F_d/N)}\times\partial B}(w_n)}{n} = \lim_{n\to\infty} \frac{H_{(\mathbb{Z}^d)^B\times\partial B}\left(\widetilde{\phi}(w_n)\right)}{n} = 0.$$

Hence, $\mathbb{Z}^{\mathrm{Edges}(F_d/N)} \times \partial B$ with the corresponding hitting measure is the Poisson boundary of $(F_d/[N,N],\mu)$.

If we furthermore suppose that μ has a finite first moment, then $\widetilde{\phi}_*\mu$ also has a finite first moment. Then Theorem 5.1.6 together with the conditional entropy criterion imply that $\lim_{n\to\infty} \frac{H_{(\mathbb{Z}^d)^B}\left(\widetilde{\phi}(w_n)\right)}{n} = 0$. Finally, we use Equation (5.13) from Lemma 5.6.10 to obtain

$$\lim_{n\to\infty} \frac{H_{\mathbb{Z}^{\mathrm{Edges}(F_d/N)}}(w_n)}{n} = \lim_{n\to\infty} \frac{H_{(\mathbb{Z}^d)^B}\left(\widetilde{\phi}(w_n)\right)}{n} = 0,$$

so that $\mathbb{Z}^{\mathrm{Edges}(F_d/N)}$ endowed with the corresponding hitting measure is the Poisson boundary of $(F_d/[N,N],\mu)$. □

Proof of Corollary 5.1.9. Consider the free solvable group $S_{d,k}$ of rank $d \geq 2$ and derived length $k \geq 2$. Let μ be an adapted probability measure on $S_{d,k}$, and suppose that μ has a finite first moment. Denote by $\widetilde{\phi} : S_{d,k} \to \mathbb{Z}^d \wr S_{d,k-1}$ the Magnus embedding.

Suppose that μ induces a transient random walk on $S_{d,k-1}$, via the projection to $\widetilde{\pi} : S_{d,k} \to S_{d,k-1}$. Note that this is always the case if $d \geq 3$ or $k \geq 3$. Indeed, in such cases the group $S_{d,k-1}$ grows at least cubically and hence any adapted measure will induce a transient random walk [Varopoulos, 1986] (see also [Woess, 2000, Theorem 3.24]). Then, it follows from the second item of Corollary 5.1.8 that the Poisson boundary of $(\mathbb{Z}^d \wr S_{d,k-1}, \widetilde{\phi}_*\mu)$ is $(\mathbb{Z}^d)^{S_{d,k-1}}$, endowed with the corresponding hitting measure.

Suppose now that μ induces a recurrent random walk on $S_{d,k-1}$. As we remarked above, since μ is adapted, this can only happen when $d = k = 2$. In this case, the Poisson boundary of the μ-random walk on $S_{2,2}$ coincides with the Poisson boundary of the $\widetilde{\phi}_*\mu$-random walk on $\mathbb{Z}^2 \wr \mathbb{Z}^2$. Since $\widetilde{\phi}_*\mu$ induces a recurrent random walk on $\mathbb{Z}^2$, the aforementioned Poisson boundary is isomorphic to the Poisson boundary of the induced random walk on the subgroup $\bigoplus_{\mathbb{Z}^2} \mathbb{Z}^2$ (see [Furstenberg, 1971, Lemma 4.2]). Since this subgroup is abelian, we conclude the triviality of the Poisson boundary. □

Articles

Eduardo Silva. "*Dead ends on wreath products and lamplighter groups*". International Journal of Algebra and Computation Vol. 33, No. 08, pp. 1489-1530 (2023).
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Eduardo Silva. "*The Poisson boundary of lampshuffler groups*". Submitted (2023).
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Joshua Frisch, Eduardo Silva. "*The Poisson boundary of wreath products*". Submitted (2023).
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ABSTRACT

In this thesis, we investigate combinatorial, geometric, and probabilistic properties of wreath products and other group extensions. The work is divided into the following two parts.

Non-extendable geodesics in Cayley graphs. We study the property of having *unbounded depth* in Cayley graphs of wreath products. That is, whether there exist elements at arbitrarily large distance from other elements of larger word length. We prove that for any finite group A and any finitely generated group B, the wreath product $A \wr B$ admits a standard generating set with unbounded depth. If B is abelian, then the above is true for every standard generating set. This generalizes the case $B = \mathbb{Z}$, due to Cleary and Taback. When $B = H * K$ for two finite groups H and K, we characterize which standard generators of $A \wr B$ have unbounded depth in terms of a geometrical constant related to the Cayley graphs of H and K.

Random walks and Poisson boundaries of groups. First, we study random walks on the lampshuffler group $\mathrm{FSym}(H) \rtimes H$, where H is a finitely generated group and $\mathrm{FSym}(H)$ is the group of finitary permutations of H. We show that for any step distribution μ with a finite first moment that induces a transient random walk on H, the permutation coordinate of the random walk almost surely stabilizes pointwise to a limit function. Our main result states that for $H = \mathbb{Z}$, the Poisson boundary of the random walk $(\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}, \mu)$ is equal to the space of limit functions endowed with the hitting measure. Our result provides new examples of completely described non-trivial Poisson boundaries of elementary amenable groups.

Next, in collaboration with Joshua Frisch, we completely describe the Poisson boundary of the wreath product $A \wr B$ of countable groups A and B, for all probability measures μ with finite entropy and such that the lamp configurations stabilize almost surely along sample paths. If in addition the projection of μ to B is Liouville, we prove that the Poisson boundary of $(A \wr B, \mu)$ coincides with the space of limit lamp configurations, endowed with the corresponding hitting measure. This improves earlier results by Lyons-Peres and, in particular, we answer an open question asked by Kaimanovich and Lyons-Peres for $B = \mathbb{Z}^d$, $d \geq 3$, and measures μ with a finite first moment.

KEYWORDS

wreath products, random walks, Poisson boundary, dead ends

RÉSUMÉ

Dans cette thèse, nous étudions des propriétés combinatoires, géométriques, et probabilistes des produits en couronne ainsi que d'autres extensions de groupes. Ce travail est divisé en deux parties.

Géodésiques non extensibles dans des graphes de Cayley. Nous étudions la propriété d'avoir *profondeur non-bornée* dans les graphes de Cayley des produits en couronne. En d'autres termes, nous cherchons à savoir s'il existent des éléments situés à une distance arbitrairement grande d'autres éléments ayant une longueur de mot plus grande. Nous prouvons que pour tout groupe fini A et tout groupe de type fini B, le produit en couronne $A \wr B$ admet un ensemble de générateurs standard avec une profondeur non bornée. Si B est abélien, alors ce qui précède est vrai pour tout ensemble générateur standard. Ceci généralise le cas $B = \mathbb{Z}$, dû à Cleary et Taback. Lorsque $B = H * K$ pour deux groupes finis H et K, nous caractérisons quels générateurs standards de $A \wr B$ ont une profondeur non bornée en termes d'une constante géométrique liée aux graphes de Cayley de H et K.

Marches aléatoires et bords de Poisson des groupes. D'abord, nous étudions des marches aléatoires sur le groupe $\mathrm{FSym}(H) \rtimes H$, où H est un groupe de type fini et $\mathrm{FSym}(H)$ est le groupe des permutations de support fini de H. Nous montrons que pour toute distribution μ des incréments avec un premier moment fini induisant une marche aléatoire transiente sur H, la coordonnée de permutation de la marche aléatoire se stabilise presque sûrement. Notre résultat principal affirme que le bord de Poisson de la marche aléatoire $(\mathrm{FSym}(\mathbb{Z}) \rtimes \mathbb{Z}, \mu)$ est égal à l'espace des fonctions limites doté de la mesure harmonique correspondante. Cela fournit de nouveaux exemples de bords de Poisson non-triviaux complètement décrits pour un groupe élémentairement moyennable.

Ensuite, en collaboration avec Joshua Frisch, nous décrivons complètement le bord de Poisson du produit en couronne $A \wr B$ des groupes dénombrables A et B, pour toutes les mesures de probabilité μ d'entropie finie et telles que les configurations de lampe se stabilisent presque sûrement. Si en plus la projection de μ sur B a la propriété de Liouville, le bord de Poisson de $(A \wr B, \mu)$ est égal à l'espace de configurations de lampes limites, doté de la mesure harmonique. Cela généralise des résultats précédents de Lyons-Peres pour $d \geq 3$ et, en particulier, nous répondons à une question ouverte posée par Kaimanovich et Lyons-Peres pour $B = \mathbb{Z}^d$, $d \geq 3$, et des mesures μ avec un premier moment fini.

MOTS CLÉS

produits en couronne, marches aléatoires, bord de Poisson, impasses, culs-de-sac